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ADDITION TO PROF. R. S. BALL'S PAPER, "NOTE ON A TRANSFORMATION OF LAGRANGE'S EQUATIONS OF MOTION IN GENERALISED COORDINATES, WHICH IS CONVENIENT IN PHYSICAL ASTRONOMY."

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THE formulæ may be established in a somewhat different way, as follows:—

Consider the masses M_1, M_2, ...

Let X_1, Y_1, Z_1 be the coordinates (in reference to a fixed origin and axes) of the C.G. of M_1;

x_1, y_1, z_1 the coordinates (in reference to a parallel set of axes through the C.G. of M_1) of an element m_1 of the mass M_1, and similarly for the masses M_2, ...; the coordinates (X_1, Y_1, Z_1), (X_2, Y_2, Z_2), ... all belonging to the same origin and axes;

And let $\dot{X}_1$, &c. denote the derived functions $\dfrac{dX_1}{dt}$, &c.

We have

$$\begin{aligned} T = &\quad S\tfrac{1}{2}m_1\left[(\dot{X}_1+\dot{x}_1)^2+(\dot{Y}_1+\dot{y}_1)^2+(\dot{Z}_1+\dot{z}_1)^2\right] \\ &+S\tfrac{1}{2}m_2\left[(\dot{X}_2+\dot{x}_2)^2+(\dot{Y}_2+\dot{y}_2)^2+(\dot{Z}_2+\dot{z}_2)^2\right] \\ &\quad\vdots \end{aligned}$$

or since $Sm_1x_1 = 0$, &c., and therefore also $Sm_1\dot{x}_1 = 0$, &c., this is

$$\begin{aligned} T = &\quad \tfrac{1}{2}M_1(\dot{X}_1^2+\dot{Y}_1^2+\dot{Z}_1^2) &&+S\tfrac{1}{2}m_1(\dot{x}_1^2+\dot{y}_1^2+\dot{z}_1^2) \\ &+\tfrac{1}{2}M_2(\dot{X}_2^2+\dot{Y}_2^2+\dot{Z}_2^2) &&+S\tfrac{1}{2}m_2(\dot{x}_2^2+\dot{y}_2^2+\dot{z}_2^2) \\ &\quad\vdots &&\quad\vdots \end{aligned}$$

Write u, v, w for the coordinates of the C.G. of the whole system: then

$$M_1X_1 + M_2X_2 + \ldots = (M_1 + M_2 \ldots)\, u,$$
$$M_1Y_1 + M_2Y_2 + \ldots = (M_1 + M_2 \ldots)\, v,$$
$$M_1Z_1 \;+ M_2Z_2 + \ldots = (M_1 + M_2 \ldots)\, w;$$

and thence

$$M_1\dot{X}_1 + M_2\dot{X}_2 + \ldots = (M_1 + M_2 \ldots)\, \dot{u},$$
$$M_1\dot{Y}_1 + M_2\dot{Y}_2 + \ldots = (M_1 + M_2 \ldots)\, \dot{v},$$
$$M_1\dot{Z}_1 \;+ M_2\dot{Z}_2 + \ldots = (M_1 + M_2 \ldots)\, \dot{w};$$

and thence

$$\begin{aligned} T - \tfrac{1}{2}(M_1 + M_2 + \ldots)(\dot{u}^2 + \dot{v}^2 + \dot{w}^2) & \\ = \frac{1}{M_1 + M_2 \ldots}\, & \{M_1M_2[(\dot{X}_1 - \dot{X}_2)^2 + (\dot{Y}_1 - \dot{Y}_2)^2 + (\dot{Z}_1 - \dot{Z}_2)^2]\} \\ & \qquad \vdots \\ + & S\tfrac{1}{2}m_1(\dot{x}_1^2 + \dot{y}_1^2 + \dot{z}_1^2) \\ + & S\tfrac{1}{2}m_2(\dot{x}_2^2 + \dot{y}_2^2 + \dot{z}_2^2) \\ & \quad \vdots \end{aligned}$$

or, representing the function on the right-hand side by T', this is

$$T = \tfrac{1}{2}(M_1 + M_2 + \ldots)(\dot{u}_2 + \dot{v}_2 + \dot{w}_2) + T'\ldots, = T_0 + T'.$$

Suppose the positions are determined by means of the $6n$ coordinates $((q))$; the equations of motion are each of them of the form

$$\frac{d}{dt}\cdot\frac{dT_0}{d\dot{q}} - \frac{dT_0}{dq} + \frac{d}{dt}\cdot\frac{dT'}{d\dot{q}} - \frac{dT'}{dq} = -\frac{dV}{dq}.$$

But these admit of further reduction; the part in T_0 depends upon three terms, such as

$$\frac{d}{dt}\left(\dot{u}\frac{d\dot{u}}{d\dot{q}}\right) - \dot{u}\frac{d\dot{u}}{dq}, \quad = \frac{d\dot{u}}{dt}\frac{d\dot{u}}{d\dot{q}} + \dot{u}\left(\frac{d}{dt}\frac{d\dot{u}}{d\dot{q}} - \frac{d\dot{u}}{dq}\right).$$

But we have u a function of $((q))$, and thence

$$\frac{d\dot{u}}{d\dot{q}} = \frac{du}{dq}, \text{ or } \frac{d}{dt}\frac{d\dot{u}}{d\dot{q}} - \frac{d\dot{u}}{dq}, \; = \frac{d}{dt}\frac{du}{dq} - \frac{d\dot{u}}{dq}, \; = 0,$$

or the term is simply

$$= \frac{d\dot{u}}{dt}\frac{d\dot{u}}{dq}.$$

The equation thus becomes

$$(M_1 + M_2\ldots)\left(\frac{d\dot{u}}{dt}\frac{d\dot{u}}{d\dot{q}} + \frac{d\dot{v}}{dt}\frac{d\dot{v}}{d\dot{q}} + \frac{d\dot{w}}{dt}\frac{d\dot{w}}{d\dot{q}}\right) + \frac{d}{dt}\frac{dT'}{d\dot{q}} - \frac{dT'}{dq} = -\frac{dV}{dq}.$$

Suppose now that T', V are functions of $6n-3$ out of the $6n$ coordinates $((q))$, and of the differential coefficients $\dot{q}$ of the same $6n-3$ coordinates, but are independent of the remaining three coordinates and of their differential coefficients; then, first, if q denotes any one of the three coordinates, the equation becomes

$$\frac{d\dot{u}}{dt}\frac{d\dot{u}}{d\dot{q}}+\frac{d\dot{v}}{dt}\frac{d\dot{v}}{d\dot{q}}+\frac{d\dot{w}}{dt}\frac{d\dot{w}}{d\dot{q}}=0;$$

or, better,

$$\frac{d\dot{u}}{dt}\frac{du}{dq}+\frac{d\dot{v}}{dt}\frac{dv}{dq}+\frac{d\dot{w}}{dt}\frac{dw}{dq}=0;$$

and the three equations of this form give

$$\frac{d\dot{u}}{dt}=0,\quad \frac{d\dot{v}}{dt}=0,\quad \frac{d\dot{w}}{dt}=0,$$

viz. these are the equations for the conservation of the motion of the centre of gravity.

And this being so, then, if q now denotes any one of the $6n-3$ coordinates, each of the remaining equations assumes the form

$$\frac{d}{dt}\cdot\frac{dT'}{d\dot{q}}-\frac{dT'}{dq}=-\frac{dV}{dq},$$

viz. we have thus $6n-3$ equations for the relative motion of the bodies of the system.