587.

A SMITH'S PRIZE DISSERTATION.

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WRITE a dissertation:

On the general equation of virtual velocities.

Discuss the principles of Lagrange's proof of it and employ it [*the general equation*] *to demonstrate the Parallelogram of Forces.*

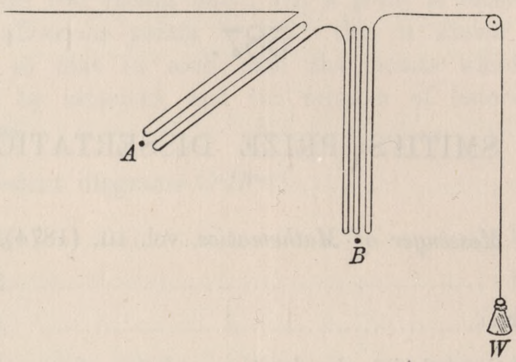
Imagine a system of particles connected with each other in any manner and subject to any geometrical conditions, for instance, two particles may be such that their distance is invariable, a particle may be restricted to move on a given surface, &c. And let each particle be acted upon by a force [this includes the case of several forces acting on the same particle, since we have only to imagine coincident particles each acted upon by a single force]. Imagine that the system has given to it any indefinitely small displacement consistent with the mutual connexions and geometrical conditions; and suppose that for any particular particle the force acting on it is P, and the displacement in the direction of the force (that is, the actual displacement multiplied into the cosine of the angle included between its direction and that of the force P) is $=\delta p$. Then δp is called the virtual velocity of the particle, and the principle of virtual velocities asserts that the sum of the products $P\delta p$, taken for all the particles of the system, and for any displacement consistent as above, is $=0$; say that we have

$$\Sigma P\delta p = 0.$$

This is also the general equation of virtual velocities: as to the mode of using it, observe that the displacements δp are not arbitrary quantities, but are in virtue of the mutual connexions and other geometrical conditions connected together by certain linear relations; or, what is the same thing, they are linear functions of certain independent arbitrary quantities δu. Substituting for δp their expressions in terms of δu

we have $\Sigma P\delta p = \Sigma U\delta u$, where the several expressions U are each of them a linear function of the forces P, and where on the right hand Σ refers to the several quantities δu; and the resulting equation is $\Sigma U\delta u = 0$; viz. since the quantities δu are independent, the equation divides itself into a set of equations $U_1 = 0$, $U_2 = 0, \ldots$ which are the equations of equilibrium of the system.

Lagrange imagines the forces produced by means of a weight W at the extremity of a string passing over a set of pulleys, as shown in the figure, viz. assuming the forces commensurable and equal to mW, nW, &c., we must have m strings at A,



n strings at B, and so on. Suppose any indefinitely small displacement given to the system; each string at A is shortened by δp, or the m strings at A by $m\delta p$; and the like for the other particles at B, &c.; hence, if $m\delta p + n\delta q + \ldots, = \frac{1}{W}(P\delta p + Q\delta q + \ldots)$, be positive, the weight W will descend through the space

$$\frac{1}{W}(P\delta p + Q\delta q + \ldots).$$

Now, in order that the system may be in equilibrium, W must be in its lowest position; or, what is the same thing, if there is any displacement allowing W to descend, W will descend, causing such displacement, and the original position is not a position of equilibrium. That is, if the system be in equilibrium, the sum $\Sigma P\delta p$ cannot be positive.

But it cannot be negative; since, if for any particular values of δp the sum $\Sigma P\delta p$ is negative, then reversing the directions of the several displacements, that is, giving to the several displacements δp the same values with opposite signs, then the sum $\Sigma P\delta p$ will be positive; and we *assume* that it is possible thus to reverse the directions of the several displacements. Hence, if the system be in a position of equilibrium, we cannot have $\Sigma P\delta p$ either positive or negative; that is, we obtain as the condition of equilibrium $\Sigma P\delta p = 0$.

The above is Lagrange's reasoning, and it seems completely unobjectionable. As regards the reversal of the directions of the displacements, observe that we consider such conditions as a condition that the particle shall be always *in* a given plane, but exclude the condition that the particle shall lie *on* a given plane, i.e. that it shall be at liberty to move in one direction (but not in the opposite direction) off from the plane. But the pulley-proof is equally applicable to a case of this kind. Thus, imagine a particle resting on a horizontal plane, and let z be measured vertically downwards, x and y horizontally. Suppose the particle acted on by the forces X, Y, Z, and replacing these by a weight W as above, the condition of equilibrium is, that

$$X\delta x + Y\delta y + Z\delta z$$

shall not be positive. We may have δx and δy, each positive or negative; whence the conditions $X = 0$ and $Y = 0$. But δz is negative; hence the required condition is satisfied if only Z is positive; that is, if the vertical force acts downwards. Clearly this is right, for if it acted upwards it would lift the particle from the plane. The case considered by Lagrange is where the particle is always *in* the plane; here $\delta z = 0$, and there is no condition as to the force Z.

The only omission in Lagrange's proof is, that he does not expressly consider the case of unstable equilibrium, where the weight W is at a position, not of minimum, but of maximum altitude. In such a case, however, the sum $\Sigma P\delta p$ is still $= 0$, taking account (as the proof does) of the displacements considered as infinitesimals of the first order; although taking account of higher powers, the sum $\Sigma P\delta p$ would have a positive value. An explanation as to this point might properly have been added to make the proof "refutation-tight," but the proof is not really in defect.

P.S. Lagrange excludes tacitly, not expressly, the case where the direction of a displacement is not reversible; he observes that the various displacements δp, when not arbitrary, are connected only by linear equations; and "par conséquent les valeurs de toutes ces quantités seront toujours telles qu'elles pourront changer de signe à la fois." The point was brought out more fully by Ostrogradsky, but I think there is no ground for the view that it was not brought out with sufficient clearness by Lagrange himself.

Parallelogram of forces.

Let P, Q, R be the forces, α, β, γ their inclinations to any line; then taking δs the displacement in the direction of this line, the displacements in the directions of the forces are $\delta s \cos\alpha$, $\delta s \cos\beta$, $\delta s \cos\gamma$, and the equation $\Sigma P\delta p = 0$ assumes the form

$$(P\cos\alpha + Q\cos\beta + R\cos\gamma)\,\delta s = 0,$$

that is, we have

$$P\cos\alpha + Q\cos\beta + R\cos\gamma = 0,$$

viz. this equation holds whatever be the fixed line to which the forces are referred. It is easy to see that, supposing it to hold in regard to any two lines, it will hold generally, and that the relation in question is thus equivalent to two independent conditions; and forming these we may obtain from them the theorem of the parallelogram of forces.

But to obtain this more directly, take A, B, C for the angles between the forces Q and R, R and P, P and Q respectively, then $A+B+C=2\pi$, and thence

$$\alpha=\alpha,$$

$$\beta=\alpha+C,$$

$$\gamma=\alpha+C+A=\alpha+2\pi-B,$$

whence writing $\alpha=\frac{1}{2}\pi$, or taking the line of displacement at right angles to the force P, we have

$$\alpha=\tfrac{1}{2}\pi,\quad \beta=\tfrac{1}{2}\pi+C,\quad \gamma=2\pi+\tfrac{1}{2}\pi-B,$$

and the equation becomes $0P-Q\sin C+R\sin B=0$, that is, $Q:R=\sin B:\sin C$; and similarly $R:P=\sin C:\sin A$, that is,

$$P:Q:R=\sin A:\sin B:\sin C,$$

equations which in fact express that each force is equal and opposite to the diagonal of the parallelogram formed by the other two forces.