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PROBLEM.

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It is required to place two given tetrahedra in perspective; or, what is the same thing, the tetrahedra being $ABCD$, $A'B'C'D'$ respectively, to place these so that the lines AA', BB', CC', DD' may meet in a point O.

The following considerations present themselves in regard to the solution of this problem. Take the tetrahedron $ABCD$ to be given in position, and the point O at pleasure; then drawing the lines OA, OB, OC, OD, we may in a determinate number of ways (viz. in 16 different ways) place the tetrahedron $A'B'C'D'$ in such manner that the summits A', B', C' shall be in the lines OA, OB, OC respectively. But the summit D' will then not be in general in the line OD; and in order that it may be so, a two-fold condition must be satisfied by the point O; viz. the locus of this point must be a certain curve in space.

Or again, we may look at the question thus: we have to place a point O in relation to the tetrahedron $ABCD$, and a point O' in relation to the tetrahedron $A'B'C'D'$, in such manner that the edges of the first tetrahedron subtend at O the same angles that the edges of the second tetrahedron subtend at O'; for this being done, then considering O' as rigidly connected with $A'B'C'D'$, we may move the figure $O'A'B'C'D'$ so that O' shall coincide with O, and the lines $O'A'$, $O'B'$, $O'C'$, $O'D'$ with OA, OB, OC, OD respectively. Take a, b, c, f, g, h, for the sides of the tetrahedron $ABCD$ ($BC, CA, AB, AD, BD, CD = a, b, c, f, g, h$ respectively), and take also x, y, z, w for the distances OA, OB, OC, OD respectively; and let $a', b', c', f', g', h', x', y', z', w'$ have the like significations in regard to the tetrahedron $A'B'C'D'$ and the point O', and write

$$\frac{y^2+z^2-a^2}{2yz},\quad \frac{z^2+x^2-b^2}{2zx},\quad \frac{x^2+y^2-c^2}{2xy},\quad \frac{x^2+w^2-f^2}{2xw},\quad \frac{y^2+w^2-g^2}{2yw},\quad \frac{z^2+w^2-h^2}{2zw},$$

$$=A,\quad B,\quad C,\quad F,\quad G,\quad H,$$

respectively; and the like as regards the accented letters. Then A, B, C, F, G, H are the cosines of the angles which the edges of the tetrahedron $ABCD$ subtend at O; they are consequently the cosines of the six sides of the spherical quadrangle obtained by the projection of $ABCD$ on a sphere centre O; and they are therefore not independent, but are connected by a single equation; substituting for A, B, C, F, G, H their values, we have a relation between $a, b, c, f, g, h, x, y, z, w$; viz. this is the relation which connects the ten distances of the five points in space O, A, B, C, D (and which relation was originally obtained by Carnot in this very manner). There is of course the like relation between the accented letters.

The conditions as to the two tetrahedra are

$$A = A',\ B = B',\ C = C',\ F = F',\ G = G',\ H = H',$$

which, attending to the relations just referred to and therefore regarding w as a given function of x, y, z, and w' as a given function of x', y', z', are equivalent to five equations (or rather to a five-fold relation); the elimination of x', y', z' from the five-fold relation gives therefore a two-fold relation between x, y, z, that is, between the distances OA, OB, OC; or the locus of O is as before a curve in space.

The conditions may be written:

$$\begin{aligned} y'^2 + z'^2 - 2Ay'z' &= a'^2, & x'^2 + w'^2 - 2Fx'w' &= f'^2, \\ z'^2 + x'^2 - 2Bz'x' &= b'^2, & y'^2 + w'^2 - 2Gy'w' &= g'^2, \\ x'^2 + y'^2 - 2Cx'y' &= c'^2, & z'^2 + w'^2 - 2Hz'w' &= h'^2; \end{aligned}$$

whence eliminating x', y', z', w', and in the result regarding A, B, C, F, G, H as given functions of x, y, z, w, we have between x, y, z, and w a three-fold relation determining w as a function of x, y, z, and establishing besides a two-fold relation between x, y, z.

As a particular case: One of the tetrahedra may degenerate into a plane quadrangle, and we have then the problem: a given plane quadrangle $ABCD$ being assumed to be the perspective representation of a given tetrahedron $A'B'C'D'$, it is required to determine the positions in space of this tetrahedron and of the point of sight O.

A generalisation of the original problem is as follows: determine the two-fold relation which must subsist between the $4 \times 6, = 24$ coordinates of four lines, in order that it may be possible to place in the tetrad of lines a given tetrahedron; that is, to place in the four lines respectively the four summits of the given tetrahedron. It may be remarked that considering three of the four lines as given, say these lines are the loci of the summits A, B, C respectively, we can in 16 different ways place in these lines respectively the three summits, and for each of these there are two positions of the summit D; there are consequently 32 positions of D; and the two-fold relation, considered as a relation between the six coordinates of the remaining line, must in effect express that this line passes through some one of the 32 points.