589.

ON RESIDUATION IN REGARD TO A CUBIC CURVE.

[From the *Messenger of Mathematics*, vol. III. (1874), pp. 62—65.]

THE following investigation of Prof. Sylvester's theory of Residuation may be compared with that given in Salmon's *Higher Plane Curves*, 2nd Edition (1873), pp. 133—137:

If the intersections of a cubic curve U_3 with any other curve V_n are divided in any manner into two systems of points, then each of these systems is said to be the residue of the other; and, in like manner, if starting with a given system of points on a cubic curve we draw through them a curve of any order V_n, then the remaining intersections of this curve with the cubic constitute a residue of the original system of points.

If the number of points in the original system is $=3p$, then the number of points in the residual system is $=3q$; and if we again take the residue, and so on indefinitely, the number of points in each residue will be $\equiv 0$ (Mod. 3); viz. we can never in this way arrive at a single point. But if the number of points in the original system be $3p\pm 1$, then that in the residual system will be $3q\mp 1$; and we may in an infinity of different ways arrive at a residue consisting of a single point; or say at a "residual point," viz. after an odd number of steps if the original number of points is $=3p-1$, but after an even number of steps if the original number of points is $=3p+1$. But starting from a given system of points on a given cubic curve, the residual point, however it is arrived at, will be one and the same point; this is Prof. Sylvester's theorem of the residuation of a cubic curve. For instance, starting with two given points on the cubic curve, the line joining these meets the curve in a third point, which is the residual point; any other process leading to a residual point must lead to the same point. Thus if through the 2 points we draw a conic, meeting the cubic besides in 4 points; through these a conic meeting the cubic besides

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in 2 points; and through this a line meeting the cubic besides in 1 point; this will be the before-mentioned residual point.

The general proof is such as in the following example:

Take on the cubic U_3 a system of $3\kappa-2$ points, say the points α: through these a curve V_k, besides meeting the cubic in $3k-3\kappa+2$ points β: and through these a curve $P_{k-\kappa+1}$, besides meeting the cubic in a point C. And again through the $3\kappa-2$ points α a curve $W_{k'}$, besides meeting the cubic in $3k'-3\kappa+2$ points β': and through these a curve $Q_{k'-\kappa+1}$, besides meeting the cubic in a single point; this will be the point C.

The proof consists in showing that we have a curve $A_{k+k'-\kappa-2}$ such that

$$A_{k+k'-\kappa-2}\, U_3 = Q_{k'-\kappa+1}\, V_k + P_{k-\kappa+1}\, W_{k'}.$$

For this observe that

$Q_{k'-\kappa+1}$ meets $W_{k'}$ in $3k'-3\kappa+2$ points β' and besides in $k'^2-k'(\kappa+2)+3\kappa-2$ points ϵ';

$P_{k-\kappa+1}$ meets V_k in $3k-3\kappa+2$ points β and besides in $k'^2-k(\kappa+2)+3\kappa-2$ points ϵ;

$P_{k-\kappa+1}$, $Q_{k'-\kappa+1}$ meet in $(k-\kappa+1)(k'-\kappa+1)$ points C;

V_k, $W_{k'}$ meet in $3\kappa-2$ points α and $kk'-3\kappa+2$ points a;

$Q_{k'-\kappa+1}\, V_k$ and $P_{k-\kappa+1}\, W_{k'}$ meet in

$$\begin{array}{llll}
kk'-k(\kappa-1)-k'(\kappa-1)+(\kappa-1)^2 & & \text{points} & C\\
3k' & -3\kappa+2 & ,, & \beta'\\
k'^2 & -k'(\kappa+2)+3\kappa-2 & ,, & \epsilon'\\
3k & -3\kappa+2 & ,, & \beta\\
k^2-k(\kappa+2) & +3\kappa-2 & ,, & \epsilon\\
kk' & -3\kappa+2 & ,, & a\\
 & 3\kappa-2 & ,, & \alpha\\
\hline
(k+k')^2-(2\kappa-2)(k+k')+(\kappa-1)^2 & & &\\
=(k+k'-\kappa+1)^2 \text{ points.} & & &
\end{array}$$

Every $(k+k'-\kappa+1)$thic through

$$\tfrac{1}{2}(k+k'-\kappa+1)(k+k'-\kappa+4)-1$$

of these points passes through all.

Now $A_{k+k'-\kappa-2}$ may be drawn to pass through

$$\tfrac{1}{2}(k+k'-\kappa-2)(k+k'-\kappa+1)$$

of the points a.

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Hence $A_{k+k'-\kappa-2}\, U_3$ is a $(k+k'-\kappa+1)$thic through

$$\begin{array}{rll} & \tfrac{1}{2}(k+k'-\kappa-2)(k+k'-\kappa+1) & \\ = & \tfrac{1}{2}(k+k')^2 - \tfrac{1}{2}(2\kappa+1)(k+k') + \tfrac{1}{2}(\kappa^2+\kappa-2) & \text{points } a \\ & \qquad\qquad 3\kappa-2 & \quad ,, \quad \alpha \\ & \qquad 3k \qquad -3\kappa+2 & \quad ,, \quad \beta \\ & \qquad 3k' \qquad -3\kappa+2 & \quad ,, \quad \beta' \\ \hline & \tfrac{1}{2}(k+k')^2 + (-\kappa+\tfrac{5}{2})(k+k') + \tfrac{1}{2}\kappa^2 - \tfrac{5}{2}\kappa + 1 & \\ = & \tfrac{1}{2}\{(k+k')^2 + (k+k')(-2\kappa+5) + (\kappa-1)(\kappa+4) - 2\} & \\ = & \tfrac{1}{2}(k+k'-\kappa+1)(k+k'-\kappa+4) - 1 & \end{array}$$

of the points in question; and therefore through all. Whence

$$A_{k+k'-\kappa-2}\, U_3 = Q_{k'-\kappa+1}\, V_k + P_{k-\kappa+1}\, W_{k'}.$$

Also U_3 meets $Q_{k'-\kappa+1}\, V_k$ in $3(k+k'-\kappa+1)$ of the $(k+k'-\kappa+1)^2$ points, viz. these are

$$\begin{array}{rcl} 3\kappa-2 & \text{points} & \alpha, \\ 3k-3\kappa+2 & ,, & \beta, \\ 3k'-3\kappa+2 & ,, & \beta', \\ 1 & ,, & C, \end{array}$$

and $A_{k+k'-\kappa-2}$ meets $Q_{k'-\kappa+1}\, V_k$ in $(k+k'-\kappa-2)(k+k'-\kappa+1)$,

that is, in

$$(k+k')^2 + (k+k')(-2\kappa-1) + \kappa^2+\kappa-2$$

of the

$$(k+k'-\kappa+1)^2 \text{ points},$$

viz. these are

$$\begin{array}{lllll} kk' & +(k+k')(-\kappa+1) + \kappa^2 - 2\kappa & & \text{points} & C \\ k'^2 & -k'(\kappa+2) & +3\kappa-2 & ,, & \epsilon' \\ k^2 & -k(\kappa+2) & +3\kappa-2 & ,, & \epsilon \\ kk' & & -3\kappa+2 & ,, & a \\ \hline \multicolumn{3}{l}{(k+k')^2 + (k+k')(-2\kappa-1) + \kappa^2 + \kappa - 2} & \text{points.} & \end{array}$$

Hence U_3 passes through 1 of the points C, that is, through an intersection of $Q_{k'-\kappa+1}$ and $P_{k-\kappa+1}$, that is, $Q_{k'-\kappa+1}$ and $P_{k-\kappa+1}$ intersect U_3 in a common point C; which was the theorem to be proved.

In the particular case $3\kappa-2=10$, $k=k'=4$, the theorem is, given on a cubic 10 points, if through these we draw a quartic meeting the cubic besides in 2 points;

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and through these a line meeting the cubic besides in a point C; then this is a fixed point, independent of the particular quartic. And the proof is as follows: we have

U a cubic through 10 points α;

V a quartic through the 10 points, and besides meeting the cubic in 2 points β;

W a quartic through the 10 points, and besides meeting the cubic in 2 points β';

P the line joining the two points β, and besides meeting V in two points ϵ;

Q the line joining the two points β', and besides meeting W in two points ϵ'; P, Q meet in the point C;

U, V meet in the 10 points α, and besides in 6 points a;

A a conic through 5 of the points a.

Then quintics QV, PW meet in the 10 points α, 2 points β, 2 points ϵ, 2 points β', 2 points ϵ', 6 points a and 1 point C. Every quintic through 19 of these passes through the 25. But we have AU, a quintic through 5 points a, and the 10 points α, 2 points β and 2 points β'; hence AU passes through all the remaining points, or we have

$$AU = QV + PW,$$

P	passes through	β, β, ϵ, ϵ, C,	
Q	,,	β', β', ϵ', ϵ', C,	
V	,,	ϵ, ϵ, β, β,	6 points a, 10 points α,
W	,,	ϵ', ϵ', β', β',	6 points a, 10 points α,
A	,,	ϵ, ϵ, ϵ', ϵ',	6 points a,
U	,,	β, β, β', β', C,	

or, what is the same thing,

A, P	intersect in	ϵ, ϵ,	
A, Q	,,	ϵ', ϵ',	
A, V	,,	ϵ, ϵ,	6 points a,
A, W	,,	ϵ', ϵ',	6 points a,
U, P	,,	β, β, C,	
U, Q	,,	β', β', C,	
U, V	,,	β, β,	10 points α,
U, W	,,	β', β',	10 points α.

In particular U, P, Q intersect in the point C; that is, C as given by the intersection of U by the line P; and as given by the intersection of U by the line Q; is one and the same point.

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