# 592.

## ON THE MERCATOR'S-PROJECTION OF A SKEW HYPERBOLOID OF REVOLUTION.

[From the *Messenger of Mathematics*, vol. IV. (1875), pp. 17—20.]

IN a note "On the Mercator's-projection of a surface of revolution" read before the British Association, [555, (5)], I remarked that the surface might be, by its meridians and parallels, divided into infinitesimal squares; and that these would be on the map represented by two systems of parallel lines at right angles to each other, dividing the map into infinitesimal squares; and that, by taking the squares not infinitesimal but small, for instance, by considering the meridians at intervals of 10° or 5°, we might approximately construct a Mercator's-projection of the surface. But it is worth while, for the skew hyperboloid of revolution, to develop analytically the ordinary accurate solution.

Taking the equation of the surface to be

$$\frac{x^2+y^2}{a^2}-\frac{z^2}{c^2}=1,$$

{or, if as usual $a^2+c^2=a^2e^2$, then $x^2+y^2-(e^2-1)z^2=a^2$}, and writing $x=r\cos\theta,\ y=r\sin\theta$, the meridians corresponding to the several longitudes $\theta$ are in the map represented by the parallel lines $X=a\theta$, and the parallels corresponding to the several values of $z$ are in the map represented by a set of parallel lines $Z=f(z)$, the form of the function being so determined that the infinitesimal rectangles on the map are similar to those on the surface. The required relation is readily found to be

$$Z=\int\frac{\sqrt{\{(a^2+c^2)z^2+c^4\}}}{z^2+c^2}\,dz,$$

where the integral is taken from the value $z=0$.

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The substitution which first presents itself is to write herein $z=\dfrac{c^2}{\sqrt{(a^2+c^2)}}\tan\phi$; or, what is the same thing,

$$z=a\left(e-\frac{1}{e}\right)\tan\phi,$$

where observe that $a\left(e-\dfrac{1}{e}\right)$ is the distance between a focus and its corresponding directrix. The equation of the surface is satisfied by writing therein $\sqrt{(x^2+y^2)}=a\sec\psi$, $z=c\tan\psi$, and $\psi$ as thus defined is the "parametric latitude"; hence the foregoing angle $\phi$ is a deduced latitude connected with the parametric latitude $\psi$ by the equation

$$\tan\phi=\frac{c}{a\left(e-\dfrac{1}{e}\right)}\tan\psi,\ =\frac{e}{\sqrt{(e^2-1)}}\tan\psi.$$

The resulting formula in terms of $\phi$ is

$$Z=\int_0\frac{c^2\sqrt{(a^2+c^2)}\,d\phi}{\cos\phi\,(c^2+a^2\cos^2\phi)},$$

or, if we write herein $\zeta=\tan\frac{1}{2}\phi$, the formula becomes

$$Z=2c^2\sqrt{(a^2+c^2)}\int_0\frac{(1+\zeta^2)^2}{1-\zeta^2}\,\frac{d\zeta}{c^2(1+\zeta^2)^2+a^2(1-\zeta^2)^2},$$

viz. the function under the integral sign is rational. The expression is, however, complicated, and a more simple formula is obtained by using instead of $\phi$ the parametric latitude $\psi$; viz. we have $z=c\tan\psi$, and thence

$$Z=\int\frac{\sqrt{(a^2\sin^2\psi+c^2)}}{\cos\psi}\,d\psi,$$

or, putting herein

$$\sin\psi=\frac{c}{a}\,\frac{u}{\sqrt{(1-u^2)}},$$

and therefore

$$\cos^2\psi=\frac{a^2-(a^2+c^2)\,u^2}{a^2(1-u^2)},$$

and

$$a^2\sin^2\psi+c^2=\frac{c^2}{1-u^2},\quad \cos\psi\,d\psi=\frac{c}{a}\,\frac{du}{(1-u^2)^{\frac{3}{2}}},$$

the formula becomes

$$Z=c^2a\int_0\frac{du}{(1-u^2)\{a^2-(a^2+c^2)\,u^2\}},$$

or, what is the same thing,

$$=a\,(e^2-1)\int_0\frac{du}{(1-u^2)(1-e^2u^2)};$$

viz. we thus have

$$Z=\tfrac{1}{2}a\left\{\log\frac{1-u}{1+u}-e\log\frac{1-eu}{1+eu}\right\},$$

the logarithms being hyperbolic.

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As already mentioned, $u$ is connected with the parametric latitude $\psi$ by the equation

$$\sin\psi = \frac{c}{a}\frac{u}{\sqrt{(1-u^2)}}, \quad = \frac{u\sqrt{(e^2-1)}}{\sqrt{(1-u^2)}},$$

that is,

$$\sin\psi = \sqrt{(e^2-1)}\tan p, \text{ if } u = \sin p,$$

or conversely

$$u = \frac{\sin\psi}{\sqrt{(e^2-1+\sin^2\psi)}},$$

so that the point passing to infinity along the branch of the hyperbola, or $\psi$ passing from 0 to 90°, $u$ passes from 0 to $\frac{1}{e}$; and for $u=\frac{1}{e}$ the value of $Z$ becomes, as it should do, infinite. The value of $z$ in terms of $u$ is

$$z = \frac{(e^2-1)u}{\sqrt{(1-e^2u^2)}}, \text{ or conversely } u = \frac{z}{\sqrt{(c^2z^2+e^2-1)}},$$

and we have, moreover,

$$u = \frac{1}{e}\frac{2\zeta}{1+\zeta^2}, \quad = \frac{1}{e}\sin\phi, \quad = (\text{as before}) \frac{\sin\psi}{\sqrt{(e^2-1+\sin^2\psi)}}.$$

It will be recollected that, in the Mercator's-projection of the sphere, the longitude and latitude being $\theta$, $\phi$, the values of $X$, $Z$ are

$$X = a\theta, \quad Z = \log\tan\left(\frac{\pi}{4}+\tfrac{1}{2}\phi\right),$$

the logarithm being hyperbolic.

In the case of the rectangular hyperbola $a=c$, $=1$ suppose,

$$e=\sqrt{(2)}, \quad z=\tan\psi, \quad u=\frac{\sin\psi}{\sqrt{(1+\sin^2\psi)}}, \quad =\sin p, \text{ if } \sin\psi=\tan p;$$

whence

$$Z = \tfrac{1}{2}h\,.\,l\frac{\tan(45^\circ-\frac{1}{2}p)}{\tan(45^\circ+\frac{1}{2}p)} - \tfrac{1}{2}\sqrt{(2)}\,h\,.\,l\frac{\tan(22^\circ 30'-\frac{1}{2}p)}{\tan(22^\circ 30'+\frac{1}{2}p)},$$

the first term being of course

$$= h\,.\,l\tan(45^\circ-\tfrac{1}{2}p), \text{ or } -h\,.\,l\tan(45^\circ+\tfrac{1}{2}p).$$

Transforming to ordinary logarithms, this is

$$Z = \frac{1}{\sqrt{(2)}\log e}\left[-\sqrt{(2)}\log\tan(45^\circ+\tfrac{1}{2}p) + \{\log\tan(22^\circ 30'+\tfrac{1}{2}p) - \log\tan(22^\circ 30'-\tfrac{1}{2}p)\}\right],$$

say this is

$$Z = \frac{1}{\sqrt{(2)}\log e}(-A+B),$$

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where

$$A = \sqrt{(2)} \log \tan (45^\circ + \tfrac{1}{2}p),$$

$$B = \log \tan (22^\circ 30' + \tfrac{1}{2}p) - \log \tan (22^\circ 30' - \tfrac{1}{2}p).$$

Taking $\psi$ as the argument, I tabulate $z$, $= \tan \psi$, and $Z \,.\, \sqrt{(2)} \log e$, $= - A + B$, as shown in the annexed table: the last column of which gives, therefore, the positions of the several parallels of 5°, 10°, ..., 85°; the interval of 5° between two meridians is, on the same scale,

$$\sqrt{(2)} \log e \,.\, \frac{\pi}{36} = (1{\cdot}4142136)\,({\cdot}4342945)\,({\cdot}0872665), \; = {\cdot}05360;$$

viz. this is nearly equal to the arc of meridian 0° to 5°, and the table shows that the arcs 0°—5°, 5°—10°, &c. continually increase as in a Mercator's-projection of the sphere, but more rapidly; there is, however, nothing in this comparison, since the determination of latitude on the hyperboloid by the equation $z = \tan \psi$ is altogether arbitrary.

| $\psi$ | $z = \tan \psi$ | $p$ | $A$ | $B$ | $-A+B$ |
|---|---|---|---|---|---|
| 0° | 0· | 0° | 0· | 0· | 0· |
| 5 | ·08749 | 4° 58′ 51″ | ·05348 | ·10719 | ·05370 |
| 10 | ·17632 | 9 51 0 | ·10611 | ·21439 | ·10827 |
| 15 | ·26795 | 14 30 40 | ·15724 | ·32174 | ·16450 |
| 20 | ·36397 | 18 53 0 | ·20619 | ·42943 | ·22324 |
| 25 | ·46631 | 22 54 30 | ·25342 | ·53780 | ·28539 |
| 30 | ·57735 | 26 34 0 | ·29557 | ·64758 | ·35201 |
| 35 | ·70020 | 29 50 20 | ·33538 | ·75959 | ·42421 |
| 40 | ·83910 | 32 44 0 | ·37170 | ·87506 | ·50337 |
| 45 | 1·00000 | 35 15 50 | ·40446 | ·99554 | ·59108 |
| 50 | 1·19175 | 37 27 20 | ·43360 | 1·12355 | ·68995 |
| 55 | 1·42815 | 39 19 20 | ·45913 | 1·26151 | ·80238 |
| 60 | 1·73205 | 40 53 40 | ·48117 | 1·41445 | ·93328 |
| 65 | 2·14450 | 42 11 10 | ·49971 | 1·58840 | 1·08869 |
| 70 | 2·74747 | 43 13 10 | ·51478 | 1·79504 | 1·28026 |
| 75 | 3·73205 | 44 0 30 | ·52646 | 2·05524 | 1·52877 |
| 80 | 5·67128 | 44 33 40 | ·53469 | 2·41347 | 1·87877 |
| 85 | 11·43005 | 44 53 30 | ·53973 | 3·02355 | 2·48381 |
| 90 | ∞ | 45° 0′ 0″ | ·54133 | ∞ | ∞ |

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