597.

ON A DIFFERENTIAL EQUATION IN THE THEORY OF ELLIPTIC FUNCTIONS.

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THE differential equation

$$Q^2 - Q\left(k + \frac{1}{k}\right) - 3 = 3(1-k^2)\frac{dQ}{dk},$$

considered *ante*, p. 69, [594, this volume, p. 244], belongs to a class of equations transformable into linear equations of the second order, and consequently is such that, knowing a particular solution, we can obtain the general solution.

In fact, assuming

$$Q = -3(1-k^2)\,\frac{1}{z}\,\frac{dz}{dk},$$

the equation becomes

$$9(1-k^2)^2\frac{1}{z^2}\left(\frac{dz}{dk}\right)^2 + 3(1-k^2)\left(k+\frac{1}{k}\right)\frac{1}{z}\frac{dz}{dk} - 3$$
$$= 3(1-k^2)\left\{3(1-k^2)\frac{1}{z^2}\frac{dz}{dk^2} + 6k\frac{1}{z}\frac{dz}{dk} - 3(1-k^2)\frac{1}{z}\frac{d^2z}{dk^2}\right\},$$

viz. omitting the terms in $\frac{1}{z^2}\left(\frac{dz}{dk}\right)^2$ which destroy each other, and dividing by $3(1-k^2)$, this is

$$\left(k+\frac{1}{k}\right)\frac{1}{z}\,\frac{dz}{dk} - \frac{1}{1-k^2} = 6k\frac{1}{z}\,\frac{dz}{dk} - 3(1-k^2)\frac{1}{z}\,\frac{d^2z}{dk^2},$$

or finally

$$3(1-k^2)\frac{d^2z}{dk^2} + \frac{1-5k^2}{k}\,\frac{dz}{dk} - \frac{1}{1-k^2}z = 0.$$

But knowing a particular value of Q we have

$$z = \exp.\left\{-\tfrac{1}{3}\int \frac{Qdz}{1-k^2}\right\},$$

a particular value of z, and thence in the ordinary manner the general value of z, giving the general value of Q.

The solution given in my former paper may be exhibited in a more simple form by introducing, instead of k, the variable α connected with it by the equation $k^2 = \frac{\alpha^3(2+\alpha)}{1+2\alpha}$. We have in fact, *Fundamenta Nova*, p. 25, [Jacobi's *Ges. Werke*, t. I., p. 76],

$$u^8 = \alpha^3 \frac{2+\alpha}{1+2\alpha}, \quad = k^2,$$

$$v^8 = \alpha\left(\frac{2+\alpha}{1+2\alpha}\right)^3, \quad = \lambda^2,$$

viz. these expressions of u, v in terms of the parameter α, are equivalent to, and replace, the modular equation $u^4 - v^4 + 2uv(1-u^2v^2) = 0$. We thence obtain

$$u^8v^8 = \frac{\alpha^4(2+\alpha)^4}{(1+2\alpha)^4}, \quad \frac{v^8}{u^8} = \frac{(2+\alpha)^2}{\alpha^2(1+2\alpha)^2},$$

that is,

$$uv = \sqrt{(\alpha)}\sqrt{\left(\frac{2+\alpha}{1+2\alpha}\right)}, \quad \frac{v^2}{u^2} = \frac{1}{\sqrt{(\alpha)}}\sqrt{\left(\frac{2+\alpha}{1+2\alpha}\right)},$$

and the particular solution, $Q = \frac{v^2}{u^2} + 2uv$, becomes

$$Q = \frac{1}{\sqrt{(\alpha)}}\sqrt{(1+2\alpha \,.\, 2+\alpha)}, \quad = \sqrt{\left\{5 + 2\left(\alpha + \frac{1}{\alpha}\right)\right\}}.$$

Introducing into the differential equation α in place of k, this is found to be

$$Q^2 - Q\frac{\frac{1}{\alpha^2} + \alpha^2 + 2\left(\frac{1}{\alpha} + \alpha\right)}{\sqrt{\left\{5 + 2\left(\alpha + \frac{1}{\alpha}\right)\right\}}} - 3 = (1-\alpha^2)\sqrt{\left\{5 + 2\left(\alpha + \frac{1}{\alpha}\right)\right\}}\frac{dQ}{d\alpha}.$$

But from this form it at once appears that it is convenient in place of α to introduce the new variable β, $= \alpha + \frac{1}{\alpha}$; the equation thus becomes

$$Q^2 + Q\frac{2 - 2\beta - \beta^2}{\sqrt{(5+2\beta)}} - 3 = (4-\beta^2)\sqrt{(5+2\beta)}\frac{dQ}{d\beta},$$

satisfied by $Q = \sqrt{(5+2\beta)}$; or, what is the same thing, writing $5 + 2\beta = \gamma^2$, that is, $\beta = -\frac{5}{2} + \gamma^2$, the equation becomes

$$4Q^2 + \frac{Q}{\gamma}(3 + 6\gamma^2 - \gamma^4) - 12 = -(\gamma^2-1)(\gamma^2-9)\frac{dQ}{d\gamma},$$

satisfied by $Q = \gamma$.

Writing here

$$Q = \tfrac{1}{4}(\gamma^2 - 1)(\gamma^2 - 9)\frac{1}{z}\frac{dz}{d\gamma},$$

we have for z the equation

$$(\gamma^2 - 1)(\gamma^2 - 9)\frac{d^2z}{d\gamma^2} + (3\gamma^4 - 14\gamma^2 + 3)\frac{dz}{d\gamma} - \frac{48}{(\gamma^2-1)(\gamma^2-9)}z = 0,$$

satisfied by

$$z = \left(\frac{\gamma^2 - 9}{\gamma^2 - 1}\right)^{\frac{1}{4}}.$$

[In fact, this value gives

$$z = (\gamma^2 - 9)^{\frac{1}{4}}(\gamma^2 - 1)^{-\frac{1}{4}},$$

$$\frac{dz}{d\gamma} = 4\gamma(\gamma^2 - 9)^{-\frac{3}{4}}(\gamma^2 - 1)^{-\frac{5}{4}},$$

$$\frac{d^2z}{d\gamma^2} = (-12\gamma^4 + 57\gamma^2 + 36)(\gamma^2 - 9)^{-\frac{7}{4}}(\gamma^2 - 1)^{-\frac{9}{4}},$$

which verify the equation as they should do.]

Representing for a moment the differential equation by $A\dfrac{d^2z}{d\gamma^2} + B\dfrac{dz}{d\gamma} + Cz = 0$, and putting $z_1 = \left(\dfrac{\gamma^2 - 9}{\gamma^2 - 1}\right)^{\frac{1}{4}}$, then assuming $z = z_1\int y\,d\gamma$, we find

$$A\left(z_1\frac{dy}{d\gamma} + 2y\frac{dz_1}{d\gamma}\right) + Byz_1 = 0,$$

that is,

$$\frac{1}{y}\frac{dy}{d\gamma} + \frac{2}{z_1}\frac{dz_1}{d\gamma} + \frac{B}{A} = 0,$$

viz.

$$\frac{1}{y}\frac{dy}{d\gamma} + \frac{2}{z_1}\frac{dz_1}{d\gamma} + \frac{3\gamma^4 - 14\gamma^2 + 3}{(\gamma^2 - 1)(\gamma^2 - 9)} = 0,$$

or

$$\frac{1}{y}\frac{dy}{d\gamma} + \frac{2}{z_1}\frac{dz_1}{d\gamma} + 3 + \frac{1}{\gamma^2 - 1} + \frac{15}{\gamma^2 - 9} = 0\,;$$

whence, integrating

$$\log yz_1^2 + 3\gamma - \tfrac{1}{2}\log\frac{\gamma + 1}{\gamma - 1} - \tfrac{5}{2}\log\frac{\gamma + 3}{\gamma - 3} = 0,$$

that is,

$$y = e^{-3\gamma}\frac{1}{z_1^2}\left(\frac{\gamma+1}{\gamma-1}\right)^{\frac{1}{2}}\left(\frac{\gamma+3}{\gamma-3}\right)^{\frac{5}{2}}$$

$$= e^{-3\gamma}\left(\frac{\gamma - 1\,.\,\gamma + 1}{\gamma - 3\,.\,\gamma + 3}\right)^{\frac{1}{2}}\left(\frac{\gamma+1}{\gamma-1}\right)^{\frac{1}{2}}\left(\frac{\gamma+3}{\gamma-3}\right)^{\frac{5}{2}}$$

$$= \frac{(\gamma+1)(\gamma+3)^2}{(\gamma-3)^3}e^{-3\gamma}.$$

Hence, the general value of z is

$$z = K\left(\frac{\gamma^2-9}{\gamma^2-1}\right)^{\frac{1}{4}}\int_{\gamma_0}\frac{(\gamma+1)(\gamma+3)^2}{(\gamma-3)^3}e^{-3\gamma}d\gamma,$$

the constants of integration being K and γ_0, or, what is the same thing,

$$z = \left(\frac{\gamma^2-9}{\gamma^2-1}\right)^{\frac{1}{4}}\left\{C + D\int_{\infty}\frac{(\gamma+1)(\gamma+3)^2}{(\gamma-3)^3}e^{-3\gamma}d\gamma\right\},$$

the corresponding value of Q being

$$Q = \tfrac{1}{4}(\gamma^2-1)(\gamma^2-9)\frac{1}{z}\frac{dz}{d\gamma},$$

which contains the single arbitrary constant $\frac{D}{C}$; when this vanishes, we have the foregoing particular solution $Q=\gamma$.

I recall that the expression of γ is

$$\gamma = \sqrt{(5+2\beta)}, \quad = \sqrt{\left\{5+2\left(\alpha+\frac{1}{\alpha}\right)\right\}}, \quad = \frac{1}{\sqrt{(\alpha)}}\sqrt{\{(2+\alpha)(1+2\alpha)\}},$$

where α is connected with k by the relation

$$k^2 = \frac{\alpha^3(2+\alpha)}{1+2\alpha}.$$