# 598.

## NOTE ON A PROCESS OF INTEGRATION.

[From the *Messenger of Mathematics*, vol. IV. (1875), pp. 149, 150.]

I HAD occasion to consider the integral

$$\int_0^R \frac{r^{s-1}\,dr}{\{r^2+e^2\}^{\frac{1}{2}s+q}},$$

where $e$ is small in regard to $R$ and $q$ is negative. The integral is finite when $e=0$, and it might be imagined that it could be expanded in positive powers of $e$; and, assuming it to be thus expansible, that the process would simply be to expand under the integral sign in ascending powers of $e$, and integrate each term separately, so that the series would be in integer powers of $e^2$.

Take two particular cases. First, let

$$s=2, \quad q=-\tfrac{3}{2};$$

the integral is

$$\int_0^R r\sqrt{(r^2+e^2)}\,dr = \int_0^R dr\,(r^2+\tfrac{1}{2}e^2r^0-\tfrac{1}{8}e^4r^{-2}+\ldots)$$
$$=\tfrac{1}{3}R^3+\tfrac{1}{2}e^2R+\infty\, e^4+\ldots,$$

viz. the integral is not thus obtainable: the series is right as far as it goes, but the true expansion contains a term in $e^3$; and the failure of the series to give the true expansion is indicated by the appearance of infinite coefficients. In fact, the indefinite integral is $\frac{1}{3}(r^2+e^2)^{\frac{3}{2}}$; taking this between the limits, it is

$$\tfrac{1}{3}(R^2+e^2)^{\frac{3}{2}}-\tfrac{1}{3}e^3, \quad =\tfrac{1}{3}R^3+\tfrac{1}{2}e^2R+\ldots-\tfrac{1}{3}e^3.$$

Again, let $s=1$, $q=-2$; the integral is

$$\int_0^R (r^2+e^2)^{\frac{3}{2}}\,dr = \int_0^R (r^3+\tfrac{3}{2}e^2r+\tfrac{3}{8}e^4r^{-1}+\ldots)$$
$$=\tfrac{1}{4}R^4+\tfrac{3}{4}e^2R^2+\infty\, e^4+\ldots,$$

viz. the integral is not thus obtainable: the series is right as far as it goes, but the true expansion contains a term as $e^4 \log e$, and the failure is indicated by the infinite coefficients. In fact, the indefinite integral is

$$(\tfrac{1}{4}r^3 + \tfrac{5}{8}e^2 r)\sqrt{(r^2+e^2)} + \tfrac{3}{8}e^4 \log\{r + \sqrt{(r^2+e^2)}\},$$

which between the limits is

$$(\tfrac{1}{4}R^3 + \tfrac{5}{8}e^2 R)\sqrt{(R^2+e^2)} + \tfrac{3}{8}e^4 \log \frac{R+\sqrt{(R^2+e^2)}}{e},$$

$$= \tfrac{1}{4}R^4 + \tfrac{3}{4}e^2R^2 + \dots - \tfrac{3}{8}e^4 \log e.$$

In the general case, the term causing the failure is $Ke^{-2q}$ when $q$ is fractional, and $Ke^{-2q}\log e$ when $q$ is integral. As a step towards determining the entire expansion, I notice that, writing $x = \dfrac{e^2}{e^2+r^2}$ or $r = ex^{-\frac{1}{2}}(1-x)^{\frac{1}{2}}$, the value of the integral is

$$= \tfrac{1}{2}e^{-2q}\int_X^1 x^{q-1}(1-x)^{\frac{1}{2}s-1}\,dx,$$

where

$$X = \frac{e^2}{e^2+R^2}.$$