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NOTE ON THE CASSINIAN.

[From the *Messenger of Mathematics*, vol. IV. (1875), pp. 187, 188.]

A SYMMETRICAL bicircular quartic has in general on the axis two nodofoci and four ordinary foci; viz. joining a nodofocus with either of the circular points at infinity, the joining line is a tangent to the curve at the circular point (and, this being a node of the curve, the tangent has there a three-pointic intersection): and joining an ordinary focus with either of the circular points at infinity, the joining line is at some other point a tangent to the curve, viz. an ordinary tangent of two-pointic intersection. In the case of the Cassinian, each circular point at infinity is a fleflecnode (node with an inflexion on each branch); of the four ordinary foci on the axis, one coincides with one nodofocus, another with the other nodofocus, and there remain only two ordinary foci on the axis; the so-called foci of the Cassinian are in fact the nodofoci, viz. each of these points is by what precedes a nodofocus *plus* an ordinary focus, and the line from either of these points to a circular point at infinity, *quà* tangent at a fleflecnode, has there a four-pointic intersection with the curve.

The analytical proof is very easy; writing the equation under the homogeneous form

$$\{(x-az)^2+y^2\}\{(x+az)^2+y^2\}-c^4z^4=0,$$

then the so-called foci are the points $(x=az,\ y=0)$, $(x=-az,\ y=0)$; at either of these, say the first of them, the line drawn to one of the circular points at infinity is $x=az+iy$, and substituting this value in the equation of the curve we obtain $z^4=0$, viz. the line is a tangent of four-pointic intersection; this implies that there is an inflexion at the point of contact on the branch touched by the line $x=az+iy$; and there is similarly an inflexion at the point of contact on the branch touched by the line $x=-az+iy$; viz. the circular point $x=iy$, $z=0$ is a fleflecnode; and similarly the circular point $x=-iy$, $z=0$, is also a fleflecnode.

To verify that there are on the axis only two ordinary foci, we write in the equation $x = \alpha z + iy$, and determine α by the condition that the resulting equation for y (which equation, by reason that the circular point $z=0$, $x=iy$, is a node, will be a quadric equation only) shall have two equal roots; the equation is in fact

$$\{(\alpha - a)^2 z^2 + 2(\alpha - a) iyz\} \{(\alpha + a)^2 z^2 - 2(\alpha + a) iyz\} - c^2 z^4 = 0,$$

viz. throwing out the factor z^2, this is

$$(\alpha^2 - a^2) \{(\alpha - a) z + 2iy\} \{(\alpha + a) z + 2iy\} - c^4 z^2 = 0,$$

or, what is the same thing, it is

$$(\alpha^2 - a^2) \{(\alpha z + 2iy)^2 - a^2 z^2\} - c^4 z^2 = 0,$$

viz. it is

$$(2iy + \alpha z)^2 - \left(a^2 + \frac{c^4}{\alpha^2 - a^2}\right) z^2 = 0.$$

The condition in order that this may have equal roots is

$$a^2 + \frac{c^4}{\alpha^2 - a^2} = 0, \text{ that is, } \alpha^2 = a^2 - \frac{c^4}{a^2};$$

hence α has only the two values $\pm \sqrt{\left(a^2 - \frac{c^4}{a^2}\right)}$, viz. there are only two ordinary foci.