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ON THE EXPRESSION OF THE COORDINATES OF A POINT OF A QUARTIC CURVE AS FUNCTIONS OF A PARAMETER.

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THE present short Note is merely the development of a process of Prof. Sylvester's. It will be recollected that the general quartic curve has the deficiency 3 (or it is 4-cursal); the question is therefore that of the determination of the subrational* functions of a parameter which have to be considered in the theory of curves of the deficiency 3.

Taking the origin at a point of the curve, the equation is

$$(x,\ y)^4+(x,\ y)^3+(x,\ y)^2+(x,\ y)=0;$$

and writing herein $y=\lambda x$, the equation, after throwing out the factor x, becomes

$$(1,\ \lambda)^4 x^3+(1,\ \lambda)^3 x^2+(1,\ \lambda)^2 x+(1,\ \lambda)=0;$$

or, say

$$ax^3+3bx^2+3cx+d=0,$$

where we write for shortness

$$a,\ b,\ c,\ d=(1,\ \lambda)^4,\ \tfrac{1}{3}(1,\ \lambda)^3,\ \tfrac{1}{3}(1,\ \lambda)^2,\ (1,\ \lambda);$$

viz. a, b, c, d stand for functions of λ of the degrees 4, 3, 2, and 1 respectively.

The equation may be written

$$(ax+b)^3-3(b^2-ac)(ax+b)+a^2d-3abc+2b^3=0;$$

* The expression "subrational" includes irrational, but it is more extensive; if Y, X are rational functions, the same or different, of y, x respectively and Y is determined as a function of x by an equation of the form $Y=X$, then y is a subrational function of x. The notion is due to Prof. Sylvester.

viz. writing for a moment $ax+b=2\sqrt{b^2-ac}\,.\,u$, this is

$$4u^3-3u+\frac{a^2d-3abc+2b^3}{2(b^2-ac)\sqrt{b^2-ac}}=0.$$

Hence, assuming

$$-\cos\phi=\frac{a^2d-3abc+2b^3}{2(b^2-ac)\sqrt{b^2-ac}},$$

then we have $4u^3-3u-\cos\phi=0$; consequently u has the three values $\cos\frac{1}{3}\phi$, $\cos\frac{1}{3}(\phi+2\pi)$, $\cos\frac{1}{3}(\phi-2\pi)$, and we may regard $\cos\frac{1}{3}\phi$ as representing any one of these values.

We have thus $ax+b=2\sqrt{b^2-ac}\cos\frac{1}{3}\phi$, and $y=\lambda x$, giving x and y as functions of λ and ϕ, that is, of λ. But for their expression in this manner we introduce the irrationality $\sqrt{b^2-ac}$, which is of the form $\sqrt{(1,\ \lambda)^6}$, and the trisection or derivation of $\cos\frac{1}{3}\phi$ from a given value of $\cos\phi$; viz. we have, as above, $-\cos\phi$, a function of λ of the form

$$(1,\ \lambda)^9\div(1,\ \lambda)^6\sqrt{(1,\ \lambda)^6}.$$

The equation for ϕ may be expressed in the equivalent forms

$$\sin\phi=\frac{a\sqrt{-(a^2d^2+4ac^3+4b^3d-6abcd-3b^2c^2)}}{(b^2-ac)\sqrt{b^2-ac}},$$

$$-\tan\phi=\frac{a\sqrt{-(a^2d^2+4ac^3+4b^3d-6abcd-3b^2c^2)}}{a^2d-3abc+2b^3};$$

and inasmuch as we have

$$2\sqrt{b^2-ac}=-\frac{a^2d-3abc+2b^3}{(b^2-ac)\cos\phi},$$

we may, instead of

$$ax+b=2\sqrt{b^2-ac}\cos\tfrac{1}{3}\phi,$$

write

$$ax+b=-\frac{(a^2d-3abc+2b^3)\cos\frac{1}{3}\phi}{(b^2-ac)\cos\phi};$$

or, what is the same thing,

$$=\frac{-(a^2d-3abc+2b^3)}{(b^2-ac)(4\cos^2\frac{1}{3}\phi-3)}.$$

The formulæ may be simplified by introducing μ, a function of λ, determined by the equation

$$c\mu^2-2b\mu+a=0;$$

viz. this equation is

$$\tfrac{1}{3}(1,\ \lambda)^2\mu^2-\tfrac{2}{3}(1,\ \lambda)^3\mu+(1,\ \lambda)^4=0,$$

so that $(\lambda,\ \mu)$ may be regarded as coordinates of a point on a nodal quartic curve, or a quartic curve of the next inferior deficiency 2. And we then have

$$(c\mu-b)=\sqrt{b^2-ac},$$

and consequently

$$-\cos\phi = \frac{a^2d - 3abc + 2b^3}{2(c\mu - b)^3};$$

viz. $\cos\phi$ is given as a rational function of the coordinates (λ, μ); there is, as before, the trisection; and we then have

$$ax + b = 2(c\mu - b)\cos\tfrac{1}{3}\phi, \quad y = \lambda x,$$

giving x and y as functions of λ, μ, ϕ; that is, ultimately, as functions of λ. I have not succeeded in obtaining in a good geometrical form the relation between the point (x, y) on the given quartic and the point (λ, μ) on the nodal quartic.

Reverting to the expression of $\tan\phi$, it may be remarked that $a = 0$ gives the values of λ which correspond to the four points at infinity on the given quartic curve; $a^2d^2 + 4ac^3 + 4b^3d - 6abcd - 3b^2c^2 = 0$, the values corresponding to the ten tangents from the origin; and $a^2d - 3abc + 2b^3 = 0$, the values corresponding to the nine lines through the origin, which are each such that the origin is the centre of gravity of the other three points on the line.

I take the opportunity of mentioning a mechanical construction of the Cartesian. The equation $r' = -A\cos\theta - N$ represents a limaçon (which is derivable mechanically from the circle $r' = -A\cos\theta$), and if we effect the transformation $r' = r + \dfrac{B}{r}$, the new curve is $r + \dfrac{B}{r} + A\cos\theta + N = 0$; that is, $r^2 + r(A\cos\theta + N) + B = 0$, which is, in fact, the equation of a Cartesian. The assumed transformation $r' = r + \dfrac{B}{r}$ can be effected immediately by a Peaucellier cell.