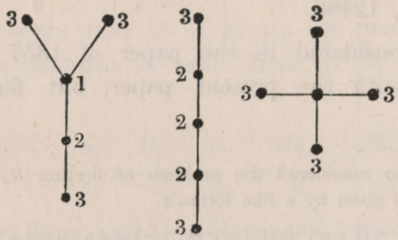
610.

ON THE ANALYTICAL FORMS CALLED TREES, WITH APPLICATION TO THE THEORY OF CHEMICAL COMBINATIONS.

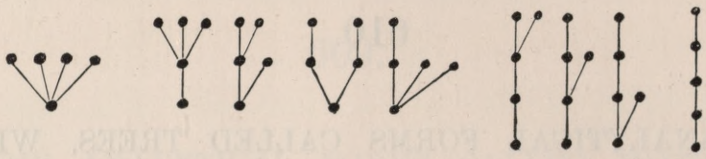
[From the *Report of the British Association for the Advancement of Science*, (1875), pp. 257—305.]

I HAVE in two papers "On the Analytical forms called Trees," *Phil. Mag.* vol. XIII. (1857), pp. 172—176, [203], and ditto, vol. XX. (1859), pp. 374—378, [247], considered this theory; and in a paper "On the Mathematical Theory of Isomers," ditto, vol. XLVII. (1874), p. 444, [586], pointed out its connexion with modern chemical theory. In particular, as regards the paraffins C_nH_{2n+2}, we have n atoms of carbon connected by $n-1$ bands, under the restriction that from each carbon-atom there proceed at most 4 bands (or, in the language of the papers first referred to, we have n knots connected by $n-1$ branches), in the form of a tree; for instance, $n=5$, such forms (and the only such forms) are



And if, under the foregoing restriction of only 4 bands from a carbon-atom, we connect with each carbon-atom the greatest possible number of hydrogen-atoms, as shown in the diagrams by the affixed numerals, we see that the number of hydrogen-atoms is 12 $(=2.5+2)$; and we have thus the representations of three different paraffins, C_5H_{12}. It should be observed that the tree-symbol of the paraffin is completely determined by means of the tree formed with the carbon-atoms, or say of the carbon-tree, and that the question of the determination of the theoretic number of the paraffins C_nH_{2n+2} is consequently that of the determination of the number of the carbon-trees of n knots, viz. the number of trees with n knots, subject to the condition that the number of branches from each knot is at most $=4$.

In the paper of 1857, which contains no application to chemical theory, the number of branches from a knot was unlimited; and, moreover, the trees were considered as issuing each from one knot taken as a root, so that, $n=5$, the trees regarded as distinct (instead of being as above only 3) were in all 9, viz. these were



which, regarded as issuing from the bottom knots, are in fact distinct; while, taking them as issuing each from a properly selected knot, they resolve themselves into the above-mentioned 3 forms. The problem considered was in fact that of the "general root-trees with n knots"—*general,* inasmuch as the number of branches from a knot was without limit; *root-trees,* inasmuch as the enumeration was made on the principle last referred to. It was found that for

knots..................	1,	2,	3,	4,	5,	6,	7,	8,......
No. of trees was ...	1,	1,	2,	4,	9,	20,	48,	115,......
=	1,	A_1,	A_2,	A_3,	A_4,	A_5,	A_6,	A_7,......;

the law being given by the equation

$$(1-x)^{-1}(1-x^2)^{-A_1}(1-x^3)^{-A_2}(1-x^4)^{-A_3}\ldots=1+A_1x+A_2x^2+A_3x^3+A_4x^4+\ldots;$$

but the next following numbers A_8, A_9, A_{10}, the correct values of which are 286, 719, 1842, were given erroneously as 306, 775, 2009. I have since calculated two more terms, A_{11}, $A_{12}=4766$, 12486.

The other questions considered in the paper of 1857 and in that of 1859 have less immediate connexion with the present paper, but for completeness I reproduce the results in a Note*.

* In the paper of 1857 I also considered the problem of finding B_r the number with r free branches, with bifurcations at least: this was given by a like formula

$$(1-x)^{-1}(1-x^2)^{-B_2}(1-x^3)^{-B_3}(1-x^4)^{-B_4}\ldots=1+x+2B_2x^2+2B_3x^3+2B_4x^4\ldots,$$

leading to

$B_r=$	1,	2,	5,	12,	33,	90, ...

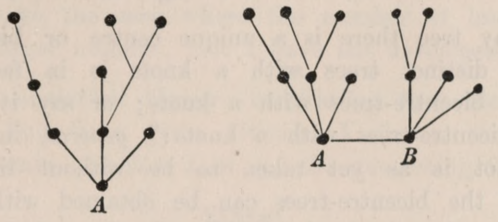
for

$r=$	2,	3,	4,	5,	6,	7, ...

In the paper of 1859, the question is to find the number of trees with a given number m of terminal knots: we have here

$$\phi m=1\,.\,2\,.\,3\ldots(m-1)\text{ coefficient of }x^{m-1}\text{ in }\frac{1}{2-e^x},$$

To count the trees on the principle first referred to, we require the notions of "centre" and "bicentre," due, I believe, to Sylvester; and to establish these we require the notions of "main branch" and "altitude": viz. in a tree, selecting any knot at pleasure as a root, the branches which issue from the root, each with all the branches that belong to it, are the main branches, and the distance of the furthest knot, measured by the number of intermediate branches, is the altitude of the main



branch. Thus in the left-hand figure, taking A as the root, there are 3 main branches of the altitudes 3, 3, 1 respectively: in the right-hand figure, taking A as the root, there are 4 main branches of the altitudes 2, 2, 1, 3 respectively; and we have then the theorem that in every tree there is either one and only one centre, or else one and only one bicentre; viz. we have (as in the left-hand figure) a centre A which is such that there issue from it two or more main branches of altitudes equal to each other and superior to those of the other main branches (if any); or else (as in the right-hand figure) a bicentre AB, viz. two contiguous knots, such that issuing from A (but not counting AB), and issuing from B (but not counting BA), we have two or more main branches, one at least from A and one at least from B, of altitudes equal to each other and superior to those of the other main branches in question (if any). The theorem, once understood, is proved without difficulty: we consider two terminal knots, the distance of which, measured by the number of intermediate branches, is greater than or equal to that of any other two terminal knots; if, as in the left-hand figure, the distance is even, then the central knot A is the centre of the tree; if, as in the right-hand figure, the distance is odd, then the two central knots AB form the bicentre of the tree.

In the former case, observe that if G, H are the two terminal knots, the distance of which is $=2\lambda$, then the distance of each from A is $=\lambda$, and there cannot be

giving the values

$\phi m=$	1,	1,	3,	13,	75,	541,	4683,	47293, ...
$m=$	1,	2,	3,	4,	5,	6,	7,	8, ...

for

But if from each non-terminal knot there ascend two and only two branches, then in this case $\phi m=$coefficient of x^{m-1} in $\frac{1-\sqrt{1-4x}}{2x}$, viz. we have the very simple form

$$\phi m=\frac{1\,.\,3\,.\,5\,...\,2m-3}{1\,.\,2\,.\,3\,...\,m}\,2^{m-1},$$

giving

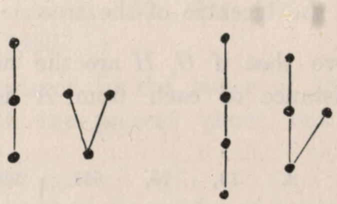
$\phi m=$	1,	1,	2,	5,	14,	42, ...
$m=$	1,	2,	3,	4,	5,	7, ...

for

any other terminal knot I, the distance of which from A is greater than λ (for, if there were, then the distance of I from G or else from H would be greater than 2λ); there cannot be any two terminal knots I, J, the distance of which is greater than 2λ; and if there are any two knots I, J, the distance of which is $=2\lambda$, then these belong to different main branches, the distance of each of them from A being $=\lambda$; whence, starting with I, J (instead of G, H), we obtain the same point A as centre. Similarly, in the latter case, there is a single bicentre AB.

Hence, since in any tree there is a unique centre or bicentre, the question of finding the number of distinct trees with n knots is in fact that of finding the number of centre- and bicentre-trees with n knots; or say it is the problem of the "general centre- and bicentre-trees with n knots:" *general*, inasmuch as the number of branches from a knot is as yet taken to be without limit; or since (as will appear) the number of the bicentre-trees can be obtained without difficulty when the problem of the root-trees is solved, the problem is that of the "general centre-trees with n knots." It will appear that the solution depends upon and is very readily derived from that of the foregoing problem of general root-trees, so that this last has to be considered, not only for its own sake, but with a view to that of the centre-trees. And in each of the two problems we doubly divide the whole system of trees according to the number of the main branches, issuing from the root or centre as the case may be, and according to the altitude of the longest main branch or branches, or say the altitude of the tree; so that the problem really is, for a given number of knots, a given number of main branches, and a given altitude, to find the number of root-trees, or (as the case may be) centre-trees.

We next introduce the restriction that the number of branches from any knot is equal to a given number at most; viz. according as this number is $=2$, 3 or 4, we have, say oxygen-trees, boron-trees*, and carbon-trees respectively; and these are, as before, root-trees or centre- or bicentre-trees, as the case may be. The case where the number is 2 presents no difficulty: in fact, if the number of knots be $=n$, then the number of root-trees is either $\frac{1}{2}(n+1)$ or $\frac{1}{2}n$; viz. $n=3$ and $n=4$, the root-trees are



and the number of centre- or bicentre-trees is always $=1$: viz. n odd, there is one centre-tree; and n even, one bicentre-tree; it is only considered as a particular case of the general theorem. The case where the number is $=3$ is analytically interesting: although there may not exist, for any 3-valent element, a series of hydrogen compounds

* I should have said nitrogen-trees; but it appears to me that nitrogen is of necessity 5-valent, as shown by the compound, Ammonium-Chloride, $=NH_4Cl$. Of course, the word boron is used simply to stand for a 3-valent element.

B_nH_{n+2} corresponding to the paraffins. The case, where the number is $=4$ or say the carbon-trees, is that which presents the chief chemical interest, as giving the paraffins C_nH_{2n+2}; and I call to mind here that the theory of the carbon-root trees is established as an analytical result for its own sake and as the foundation for the other case, but that it is the number of the carbon centre- and bicentre-trees which is the number of the paraffins.

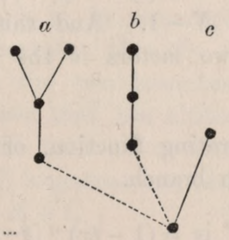
The theory extends to the case where the number of branches from a knot is at most $=5$, or $=$ any larger number; but I have not developed the formula.

I pass now to the analytical theory: considering first the case of general root-trees, we endeavour to find for a given altitude N the number of trees of a given number of knots n and main branches α, or say the generating function

$$\Sigma\Omega t^\alpha x^n,$$

where the coefficient Ω gives the number of the trees in question. And we assume that the problem is solved for the cases of the several inferior altitudes 0, 1, 2, 3,.., $N-1$.

This being so, observe that a tree of altitude N can be built up as shown in the figure, which I call the edification diagram, by combining one or more trees of altitude $N-1$ with a single tree of altitude not exceeding $N-1$; viz. in the figure, $N=3$, we have the two trees a, b, each of altitude 2, combined, as shown by the



dotted lines, with the tree c of altitude 1: the whole number of knots in the resulting tree is the sum of the number of knots on the three trees a, b, c: the number of main branches is equal to the number of the trees a, b, plus the number of main branches of the tree c. It is to be observed that the tree c may reduce itself to the tree $(\cdot)$ of one knot and of altitude zero; but each of the trees a, b, as being of the altitude $N-1$, must contain at least N knots.

Taking $N=2$ or any larger number, it is hence easy to see that the required generating function $\Sigma\Omega t^\alpha x^n$ is

$$=(1-tx^N)^{-1}(1-tx^{N+1})^{-l_1}(1-tx^{N+2})^{-l_2}\dots[t^{1\dots\infty}] \qquad \text{(first factor)},$$

$$x+(t)\,x^2+(t,\ t^2)\,x^3+(t,\ t^2,\ t^3)\,x^4+\dots \qquad \text{(second factor)}.$$

As regards the first factor, the exponents taken with reversed sign, that is, as positive, are $1=$ no. of trees, altitude $N-1$, of N knots; $l_1=$ ditto, same altitude, of $(N+1)$ knots; $l_2=$ ditto, same altitude, of $N+2$ knots, and so on; and where the symbol $[t^{1\cdots\infty}]$ denotes that, in the function or product of factors which precedes it, the terms to be taken account of are those in $t^1, t^2, t^3, \ldots$; viz. it denotes that the term in t^0, or constant term ($=1$ in fact), is to be rejected.

In the second factor, the expressions $x, (t)\,x^2, (t,\ t^2)\,x^3, \ldots$ represent, for given exponents of t, x, denoting the number of main branches and the number of knots respectively, the number of trees of altitude not exceeding $N-1$: thus $x, = 1\,t^0x^1$ represents the number of such trees, 1 knot, 0 main branch, $=1$; and so, if the value of $(t,\ t^2,\ t^3,\ t^4)\,x^5$ be $(\alpha t + \beta t^2 + \gamma t^3 + \delta t^4)\,x^5$, then for trees of an altitude not exceeding $N-1$, and of 5 knots, α represents the number of trees of 1 main branch, β that of trees of 2 main branches, γ that of trees of 3 main branches, δ that of trees of 4 main branches. It is clear that the number of trees satisfying the given conditions and of an altitude not exceeding $N-1$ is at once obtained by addition of the numbers of the trees satisfying the given conditions, and of the altitudes $0, 1, 2, \ldots, N-1$; all which numbers are taken to be known.

It is to be remarked that the first factor,

$$(1-tx^N)^{-1}\,(1-tx^{N+1})^{-l_1}\,(1-tx^{N+2})^{-l_2}\ldots[t^{1\cdots\infty}],$$

shows by its development the number of combinations of trees $a, b, \ldots$ of the altitude $N-1$; one such tree at least *must* be taken, and the symbol $[t^{1\cdots\infty}]$ gives effect to this condition: the second factor $x + (t)\,x^2 + (t,\ t^2)\,x^3 + \ldots$ shows the number of the trees c of altitude not exceeding $N-1$. And this being so, there is no difficulty in seeing how the product of the two factors is the generating function for the trees of altitude N.

In the case $N=0$, the generating function, or GF, is $=x$; viz. altitude 0, there is only the tree (·), 1 knot, 0 main branch.

$$\text{When } N=1, \text{ the } GF \text{ is } = (1-tx)^{-1}\,[t^{1\cdots\infty}]\,x, \ = tx^2 + t^2x^3 + t^3x^4 \ldots,$$

viz. altitude 1, there is 1 tree tx^2, 2 knots, 1 main branch; 1 tree t^2x^3, 3 knots, 2 main branches; and so on.

Hence $N=2$, we obtain

$$GF = (1-tx^2)^{-1}\,(1-tx^3)^{-1}\,(1-tx^4)^{-1}\ldots[t^{1\cdots\infty}]\,.\,(x + tx^2 + t^2x^3 + t^3x^4 + \ldots);$$

viz. as regards the second factor, altitude not exceeding 1, that is, $=0$ or 1, there is altitude 0, 1 tree x, and altitude 1, 1 tree tx^2, 1 tree t^2x^3, and so on. And we hence derive the GF's for the higher values $N=3, 4$, &c.: the details of the process will be afterwards more fully explained.

So far, we have considered root-trees; but referring to the last diagram, it is at once seen that the assumed root will be a centre, provided only that (instead of, it may be, only a single tree a of the altitude $N-1$), we take always two or more trees of the altitude $N-1$ to form the new tree of the altitude N. And we give effect to this condition by simply writing in place of $[t^{1\cdots\infty}]$ the new symbol $[t^{2\cdots\infty}]$, which denotes that only the terms t^2, t^3, $t^4, \ldots$ are to be taken account of; viz. that the terms in t^0 and t^1 are to be rejected. The component trees of the altitude $N-1$ are, it is to be observed, as before, root-trees; hence the second factor of the generating function is unaltered: the theorem is that for the centre-trees of altitude N we have the same generating function as for the root-trees, writing only $[t^{2\cdots\infty}]$ in place of $[t^{1\cdots\infty}]$. Or, what is the same thing, supposing that the first factor, unaffected by either symbol, is

$$= 1 + x^N(\alpha t + \beta t^2 + \ldots) + x^{N+1}(\alpha' t + \beta' t^2 + \ldots) + \ldots,$$

then, affecting it with $[t^{1\cdots\infty}]$, the value for the root-trees is

$$= x^N(\alpha t + \beta t^2 + \ldots) + x^{N+1}(\alpha' t + \beta' t^2 + \ldots) + \ldots,$$

and, affecting it with $[t^{2\cdots\infty}]$, the value for the centre-trees is

$$= x^N(\beta t^2 + \ldots) + x^{N+1}(\beta' t^2 + \ldots) + \ldots$$

It thus appears how the fundamental problem is that of the root-trees, its solution giving at once that of the centre-trees; whereas we cannot conversely solve the problem of the root-trees by means of that of the centre-trees.

As regards the bicentre-trees, it is to be remarked that, starting from a centre-tree of altitude $N+1$ with two main branches, then by simply striking out the centre, so as to convert into a single branch the two branches which issue from it, we obtain a bicentre-tree of altitude N. Observe that the altitude of a bicentre-tree is measured by that of the longest main branch from A or B, not reckoning AB or BA as a main branch. Hence the number of bicentre-trees, altitude N, is = number of centre-trees of two main branches, altitude $N+1$.

This is, in fact, the convenient formula, provided only the number of centre-trees of two main branches has been calculated up to the altitude $N+1$. But we can find independently the number of bicentre-trees of a given altitude N: the bicentre-tree is, in fact, formed by taking the two connected points A, B each as the root of a root-tree altitude N (the number of knots of the bicentre-tree being thus, it is clear, equal to the sum of the numbers of knots of the two root-trees respectively); and it is thus an easy problem of combinations to find the number of bicentre-trees of a given altitude N. Write

$$x^{N+1}(1 + \beta x + \gamma x^2 + \delta x^3 + \ldots)$$

as the generating function of the root-trees of altitude N; viz. for such trees, 1 = no. of trees with $N+1$ knots, β = no. with $N+2$ knots, and so on; then the generating function of the bicentre-trees of the same altitude N is

$$= x^{2N+2}(1 + \beta_{,} x + \gamma_{,} x^2 + \delta_{,} x^3 + \ldots),$$

where

$$\beta_{,} = \beta,$$
$$\gamma_{,} = \gamma + \tfrac{1}{2}\beta(\beta + 1),$$
$$\delta_{,} = \delta + \beta\gamma,$$
$$\epsilon_{,} = \epsilon + \beta\delta + \tfrac{1}{2}\gamma(\gamma + 1),$$
$$\zeta_{,} = \zeta + \beta\epsilon + \gamma\delta,$$

and so on; or, what is the same thing, calling the first generating function ϕx, then the second generating function is $= \frac{1}{2}\{(\phi x)^2 + \phi(x^2)\}$.

It will be noticed that the bicentre-trees are not, as were the centre-trees, divided according to the number of their main branches; they might be thus divided according to the sum of the number of the main branches issuing from the two points of the bicentre respectively; a more complete division would be according to the number of main branches issuing from the two points respectively; thus we might consider the bicentre-trees (2, 3), with 2 main branches from one point, and 3 main branches from the other point of the bicentre; but the whole theory of the bicentre-trees is comparatively easy, and I do not go into it further.

We have yet to consider the case of the limited trees where the number of branches from a knot is equal to a given number at most: to fix the ideas, say the carbon-trees, where this number is $=4$. The distinction as to root-trees and centre- and bicentre-trees is as before; and the like theory applies to the two cases respectively. Considering first the case of the root-trees, and referring to the former figure for obtaining the trees of altitude N from those of inferior altitudes, then the trees $a, b, \ldots$ of altitude $N-1$ must be each of them a carbon-tree of not more than $(4-1=)\,3$ main branches: this restriction is necessary, inasmuch as, if for any such tree the number of main branches was $=4$, then there would be from the root of such tree 4 branches *plus* the new branch shown by the dotted line, in all 5 branches; and similarly, inasmuch as there is at least one component tree a contributing one main branch, the number of main branches of the tree c must be $(4-1=)\,3$ at most: the mode of introducing these conditions will appear in the explanation of the actual formation of the generating functions (see explanation preceding Tables III., IV., &c.). The number of main branches is $=4$ at most, and the generating functions have only to be taken up to the terms in t^4; the first factor is consequently in each case affected with a symbol $[t^{1\cdots 4}]$, denoting that the only terms to be taken account of are those in t, t^2, t^3, t^4; hence as there is a factor t at least, and the whole is required only up to t^4, the second factor is in each case required only up to t^3.

As regards the centre-trees, the generating functions have here the same expressions as for the root-trees, except that, instead of the symbol $[t^{1\cdots 4}]$, we have the symbol $[t^{2\cdots 4}]$, denoting that in the first factor the only terms to be taken account of are those in t^2, t^3, t^4; hence as there is a factor t^2 at least, and the whole is required only up to t^4, the second factor is in each case required up to t^2; and we then complete the theory by obtaining the bicentre-trees. The like remarks apply of course to the boron-trees, number of branches $=3$ at most, and to the oxygen-trees, number $=2$ at most; but, as already remarked, this last case is so simple, that the general method is applied to it only for the sake of seeing what the general method becomes in such an extreme case.

We thus form the Tables, which I proceed to explain.

Table I. of general root-trees is in fact a Table of triple entry, viz. it gives for any given number of knots from 1 to 13 the number of root-trees corresponding to any given number of main branches and to any given altitude. In each compartment, that is, for any given number of knots, the totals of the columns give the number of the trees for each given altitude, and the totals of the lines give the number of the trees for each given number of main branches: the corner grand totals of these totals respectively show for each given number of knots the whole number of root-trees:—

viz. knots ...	1,	2,	3,	4,	5,	6,	7,	8,	9,	10,	11,	12,	13,
numbers are ...	1,	1,	2,	4,	9,	20,	48,	115,	286,	719,	1842,	4766,	12486,

as already mentioned: these numbers were calculated by an independent method.

Table II. of general centre- and bicentre-trees consists of a centre part and a bicentre part: the centre part is arranged precisely in the same manner as the root-table. As to the bicentre part, where it will be observed there is no division for number of main branches, the calculation of the several columns is effected by the before-mentioned formula,

$$\phi_{,}x = \tfrac{1}{2}\{(\phi x)^2 + \phi(x^2)\};$$

thus column 2, we have by Table I. (totals of column 2)

$$\phi x = x^3 + 2x^4 + 4x^5 + 6x^6 + 10x^7 + 14x^8 + 21x^9 + 29x^{10} + \dots,$$

and thence

$$\phi_{,}x = x^6 + 2x^7 + 7x^8 + 14x^9 + 32x^{10} + 58x^{11} + 110x^{12} + 187x^{13} + \dots$$

As already mentioned, each column of Table I. is calculated by means of a generating function given as a product of two factors, each of which is obtained from the columns which precede the column in question; and Table II., the centre part of it, is calculated by means of the same generating functions slightly modified: these generating functions serving for the calculation of the two Tables are given in the table entitled "Subsidiary Table for the calculation of the *GF*'s of Tables I. and II.," which immediately follows these two Tables, and will be further explained.

TABLE I.—General Root-trees.

Index x, or number of knots.	Index t, or number of main branches.	Altitude or number of column.													
		0	1	2	3	4	5	6	7	8	9	10	11	12	13
1	0	1	1												
	Total	1	**1**												
2	1		1	1											
	Total		1	**1**											
3	1			1	1										
	2		1		1										
	Total		1	1	**2**										
4	1			1	1	2									
	2			1		1									
	3		1			1									
	Total		1	2	1	**4**									
5	1			1	2	1	4								
	2			2	1		3								
	3			1			1								
	4		1				1								
	Total		1	4	3	1	**9**								
6	1			1	4	3	1	9							
	2			2	3	1		6							
	3			2	1			3							
	4			1				1							
	5		1					1							
	Total		1	6	8	4	1	**20**							
7	1			1	6	8	4	1	20						
	2			3	8	4	1		16						
	3			3	3	1			7						
	4			2	1				3						
	5			1					1						
	6		1						1						
	Total		1	10	18	13	5	1	**48**						
8	1			1	10	18	13	5	1	48					
	2			3	15	13	5	1		37					
	3			4	9	4	1			18					
	4			3	3	1				7					
	5			2	1					3					
	6			1						1					
	7		1							1					
	Total		1	14	38	36	19	6	1	**115**					
9	1			1	14	38	36	19	6	1	115				
	2			4	30	36	19	6	1		96				
	3			5	19	14	5	1			44				
	4			5	9	4	1				19				
	5			3	3	1					7				
	6			2	1						3				
	7			1							1				
	8		1								1				
	Total		1	21	76	93	61	26	7	1	**286**				

TABLE I. (*continued*).

Index x, or number of knots.	Index t, or number of main branches.	Altitude or number of column.													
		0	1	2	3	4	5	6	7	8	9	10	11	12	13
10	1			1	21	76	93	61	26	7	1	286			
	2			4	51	89	61	26	7	1		239			
	3			7	42	41	20	6	1			117			
	4			6	20	14	5	1				46			
	5			5	9	4	1					19			
	6			3	3	1						7			
	7			2	1							3			
	8			1								1			
	9		1									1			
	Total		1	29	147	225	180	94	34	8	1	719			
11	1			1	29	147	225	180	94	34	8	1	719		
	2			5	90	210	180	94	34	8	1		622		
	3			8	79	110	67	27	7	1			299		
	4			9	46	42	20	6	1				124		
	5			7	20	14	5	1					47		
	6			5	9	4	1						19		
	7			3	3	1							7		
	8			2	1								3		
	9			1									1		
	10		1										1		
	Total		1	41	277	528	498	308	136	43	9	1	1842		
12	1			1	41	277	528	498	308	136	43	9	1	1842	
	2			5	145	467	493	308	136	43	9	1		1607	
	3			10	152	278	208	101	35	8	1			793	
	4			11	91	115	68	27	7	1				320	
	5			10	47	42	20	6	1					126	
	6			7	20	14	5	1						47	
	7			5	9	4	1							19	
	8			3	3	1								7	
	9			2	1									3	
	10			1										1	
	11		1											1	
	Total		1	55	509	1198	1323	941	487	188	53	10	1	4766	
13	1			1	55	509	1198	1323	941	487	188	53	10	1	4766
	2			6	238	1012	1524	941	487	188	53	10	1		4460
	3			12	272	669	376	344	144	44	9	1			1871
	4			15	184	299	213	102	35	8	1				857
	5			13	95	116	68	27	7	1					327
	6			11	47	42	20	6	1						127
	7			7	20	14	5	1							47
	8			5	9	4	1								19
	9			3	3	1									7
	10			2	1										3
	11			1											1
	12		1												1
	Total		1	76	924	2666	3405	2744	1615	728	251	64	11	1	12486

TABLE II.—General Centre- and Bicentre-Trees.

Index x, or number of knots.	Index t, or number of main branches.	Centre-Trees. Altitude or number of column.							Centre.	Grand Total.	Bicentre.	Bicentre-Trees. Altitude.					
		0	1	2	3	4	5	6				0	1	2	3	4	5
1	0	1	1						1	1	0						
	Total	1	1														
2									0	1	1	1					
3	2		1	1													
	Total		1	1					1	1	0						
4	2																
	3		1	1													
	Total		1	1					1	2	1		1				
5	2			1	1												
	3																
	4		1		1												
	Total		1	1	2				2	3	1		1				
6	2			1	1												
	3			1	1												
	4																
	5		1		1												
	Total		1	2	3				3	6	3		2	1			
7	2			2	1	3											
	3			2		2											
	4			1		1											
	5																
	6		1			1											
	Total		1	5	1	7			7	11	4		2	2			
8	2			2	2	4											
	3			3	1	4											
	4			2		2											
	5			1		1											
	6																
	7		1			1											
	Total		1	8	3	12			12	23	11		3	7	1		
9	2			3	7	1	11										
	3			4	3		7										
	4			4	1		5										
	5			2			2										
	6			1			1										
	7																
	8		1				1										
	Total		1	14	11	1	27		27	47	20		3	14	3		

TABLE II. (*continued*).

Index x, or number of knots.	Index t, or number of main branches.	Centre-Trees. Altitude or number of column.								Centre.	Grand Total.	Bicentre.	Bicentre-Trees. Altitude.					
		0	1	2	3	4	5	6					0	1	2	3	4	5
10	2			3	14	3	20											
	3			6	11	1	18											
	4			5	3		8											
	5			4	1		5											
	6			2			2											
	7			1			1											
	8																	
	9		1				1											
	Total		1	21	29	4	55			55	106	51		4	32	14	1	
11	2			4	32	14	1	51										
	3			7	26	4		37										
	4			8	12	1		21										
	5			6	3			9										
	6			4	1			5										
	7			2				2										
	8			1				1										
	9																	
	10		1					1										
	Total		1	32	74	19	1	127		127	235	108		4	58	42	4	
12	2			4	58	42	4	108										
	3			9	63	19	1	92										
	4			10	30	4		44										
	5			9	12	1		22										
	6			6	3			9										
	7			4	1			5										
	8			2				2										
	9			1				1										
	10																	
	11		1					1										
	Total		1	45	167	66	5	284		284	551	267		5	110	128	23	1
13	2			5	110	128	23	1	267									
	3			11	132	66	5		214									
	4			14	78	20	1		113									
	5			12	31	4			47									
	6			10	12	1			23									
	7			6	3				9									
	8			4	1				5									
	9			2					2									
	10			1					1									
	11																	
	12		1						1									
	Total		1	65	367	219	29	1	682	682	1301	619		5	187	334	88	5

Subsidiary Table for *GF*'s of Tables I. and II.

Index *t*.	Index of *x*.														
	0	1	2	3	4	5	6	7	8	9	10	11	12	13	
0		1													*GF*, column 0.
*	·	−1													*GF*, column 1.
0	(1)														First factor.
1		1													
2			1												
3				1											
4					1										
5						1									
6							1								
7								1							
8									1						
9										1					
10											1				
11												1			
12													1		
0		1													Second factor.
*			−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1		*GF*, column 2.
0	(1)														First factor.
1			1	1	1	1	1	1	1	1	1	1	1		
2					1	1	2	2	3	3	4	4	5		
3							1	1	2	3	4	5	7		
4									1	1	2	3	5		
5											1	1	2		
6													1		
0		1													Second factor.
1			1												
2				1											
3					1										
4						1									
5							1								
6								1							
7									1						
8										1					
9											1				
10												1			
11													1		
12														1	
*				−1	−2	−4	−6	−10	−14	−21	−29	−41	−55	−76	*GF*, column 3.
0	(1)														First factor.
1		·	·	1	2	4	6	10	14	21	29	41	55		
2							1	2	7	14	32	58	110		
3										1	2	7	18		
4													1		
0		1													Second factor.
1			1	1	1	1	1	1	1	1	1	1	1	1	
2				1	1	2	2	3	3	4	4	5	5	6	
3					1	1	2	3	4	5	7	8	10	12	
4						1	1	2	3	5	6	9	11	15	
5							1	1	2	3	5	7	10	13	
6								1	1	2	3	5	7	11	
7									1	1	2	3	5	7	
8										1	1	2	3	5	
9											1	1	2	3	
10												1	1	2	
11													1	1	
12														1	

Subsidiary Table for *GF*'s of Tables I. and II. (*continued*).

Index *t*.	Index of *x*. 0	1	2	3	4	5	6	7	8	9	10	11	12	13	
*					-1	-3	-8	-18	-38	-76	-147	-277	-509	-924	*GF*, column 4.
0	(1)														First factor.
1				·	1	3	8	18	38	76	147	277	509		
2									1	3	14	42	128		
3													1		
0		1													Second factor.
1			1	1	2	3	5	7	11	15	22	30	42	56	
2				1	1	3	5	11	18	34	55	95	150	244	
3					1	1	3	6	13	24	49	87	162	284	
4						1	1	3	6	14	26	55	102	199	
5							1	1	3	6	14	27	57	108	
6								1	1	3	6	14	27	58	
7									1	1	3	6	14	27	
8										1	1	3	6	14	
9											1	1	3	6	
10												1	1	3	
11													1	1	
12														1	
*	·				·	-1	-4	-13	-36	-93	-225	-528	-1198	-2666	*GF*, column 5.
0	(1)														First factor.
1						1	4	13	36	93	225	528	1198		
2											1	4	23		
0		1													Second factor.
1			1	1	2	4	8	15	29	53	98	177	319	565	
2				1	1	3	6	15	31	70	144	305	617	1256	
3					1	1	3	7	17	38	90	197	440	953	
4						1	1	3	7	18	40	97	217	498	
5							1	1	3	7	18	41	99	224	
6								1	1	3	7	18	41	100	
7									1	1	3	7	18	41	
8										1	1	3	7	18	
9											1	1	3	7	
10												1	1	3	
11													1	1	
12														1	
*	·	·	·	·	·	·	-1	-5	-19	-61	-180	-498	-1323	-3405	*GF*, column 6.
0	(1)														First factor.
1							1	5	19	61	180	498	1323		
2													1		
0		1													Second factor.
1			1	1	2	4	9	19	42	89	191	402	847	1763	
2				1	1	3	6	16	36	89	205	485	1110	2780	
3					1	1	3	7	18	43	110	264	648	1329	
4						1	1	3	7	19	45	117	285	711	
5							1	1	3	7	19	46	119	292	
6								1	1	3	7	19	46	120	
7									1	1	3	7	19	46	
8										1	1	3	7	19	
9											1	1	3	7	
10												1	1	3	
11													1	1	
12														1	

Subsidiary Table for *GF*'s of Tables I. and II. (*continued*).

Index t.	Index of x.														
	0	1	2	3	4	5	6	7	8	9	10	11	12	13	
*								-1	-6	-26	-94	-308	-941	-2744	*GF*, column 7.
0	(1)														First factor.
1								1	6	26	94	308	941		
0		1													Second factor.
1			1	1	2	4	9	20	47	108	252	582	1345	3086	
2				1	1	3	6	16	37	95	231	579	1418	3721	
3					1	1	3	7	18	44	116	291	749	1673	
4						1	1	3	7	19	46	123	312	813	
5							1	1	3	7	19	47	125	319	
6								1	1	3	7	19	47	126	
7									1	1	3	7	19	47	
8										1	1	3	7	19	
9											1	1	3	7	
10												1	1	3	
11													1	1	
12														1	
*									-1	-7	-34	-136	-487	-1615	*GF*, column 8.
0	(1)														First factor.
1									1	7	34	136	487	1615	
0		1													Second factor.
1			1	1	2	4	9	20	48	114	278	676	1653	4027	
2				1	1	3	6	16	37	96	238	613	1554	4208	
3					1	1	3	7	18	44	117	298	784	1817	
4						1	1	3	7	19	46	124	319	848	
5							1	1	3	7	19	47	126	326	
6								1	1	3	7	19	47	127	
7									1	1	3	7	19	47	
8										1	1	3	7	19	
9											1	1	3	7	
10												1	1	3	
11													1	1	
12														1	
*										-1	-8	-43	-188	-728	*GF*, column 9.
0	(1)														First factor.
1										1	8	43	188	728	
0		1													Second factor.
1			1	1	2	4	9	20	48	115	285	710	1789	4514	
2				1	1	3	6	16	37	96	239	621	1597	4396	
3					1	1	3	7	18	44	117	299	792	1861	
4						1	1	3	7	19	46	124	320	856	
5							1	1	3	7	19	47	126	327	
6								1	1	3	7	19	47	127	
7									1	1	3	7	19	47	
8										1	1	3	7	19	
9											1	1	3	7	
10												1	1	3	
11													1	1	
12														1	

Subsidiary Table for *GF*'s of Tables I. and II. (*continued*).

Index t.	Index of x.														
	0	1	2	3	4	5	6	7	8	9	10	11	12	13	
*											−1	−9	−53	−251	*GF*, column 10.
0	(1)														First factor.
1											1	9	53	251	
0		1													Second factor.
1			1	1	2	4	9	20	48	115	286	718	1832	4702	
2				1	1	3	6	16	37	96	239	622	1606	4449	
3					1	1	3	7	18	44	117	299	793	1870	
4						1	1	3	7	19	46	124	320	857	
5							1	1	3	7	19	47	126	327	
6								1	1	3	7	19	47	127	
7									1	1	3	7	19	47	
8										1	1	3	7	19	
9											1	1	3	7	
10												1	1	3	
11													1	1	
12														1	
*												−1	−10	−64	*GF*, column 11.
0	(1)														First factor.
1												1	10	64	
0		1													Second factor.
1			1	1	2	4	9	20	48	115	286	719	1841	4755	
2				1	1	3	6	16	37	96	239	622	1607	4459	
3					1	1	3	7	18	44	117	299	793	1871	
4						1	1	3	7	19	46	124	320	857	
5							1	1	3	7	19	47	126	327	
6								1	1	3	7	19	47	127	
7									1	1	3	7	19	47	
8										1	1	3	7	19	
9											1	1	3	7	
10												1	1	3	
11													1	1	
12														1	
*													−1	−11	*GF*, column 12.
0	(1)														First factor.
1													1	11	
0		1													Second factor.
1			1	1	2	4	9	20	48	115	286	719	1842	4765	
2				1	1	3	6	16	37	96	239	622	1607	4460	
3					1	1	3	7	18	44	117	299	793	1871	
4						1	1	3	7	19	46	124	320	857	
5							1	1	3	7	19	47	126	327	
6								1	1	3	7	19	47	127	
7									1	1	3	7	19	47	
8										1	1	3	7	19	
9											1	1	3	7	
10												1	1	3	
11													1	1	
12														1	
*														−1	*GF*, column 13.
0	(1)														First factor.
1														1	
0		1													Second factor.
1			1	1	2	4	9	20	48	115	286	719	1842	4766	
2				1	1	3	6	16	37	96	239	622	1607	4460	
3					1	1	3	7	18	44	117	299	793	1871	
4						1	1	3	7	19	46	124	320	857	
5							1	1	3	7	19	47	126	327	
6								1	1	3	7	19	47	127	
7									1	1	3	7	19	47	
8										1	1	3	7	19	
9											1	1	3	7	
10												1	1	3	
11													1	1	
12														1	

I proceed to explain the Subsidiary Table, first in its application to Table I.

The Subsidiary Table is divided into sections, giving the *GF*'s of the successive columns of Table I., each section being given by means of the preceding columns of Table I.; for instance, that for column 3 by means of columns 0, 1, 2 of Table I.

As regards column 0, the Table shows that the *GF* is $=x$.

As regards column 1, it shows that the *GF* has a first factor,

$$(1-tx)^{-1}, =(1)+tx+t^2x^2+t^3x^3+\dots,$$

which is operated on by the symbol $[t^{1\dots\infty}]$, viz. the constant term (1) is to be rejected; and that it has a second factor, $=x$: the product of these, viz. $(tx+t^2x^2+t^3x^3+\dots)\times x$, is the required *GF*, the coefficients of which are accordingly given in column 1 of Table I.

As regards column 2, it shows that the *GF* has a first factor,

$$(1-tx^2)^{-1}(1-tx^3)^{-1}(1-tx^4)^{-1}\dots,$$

where the indices $-1, -1, -1, \dots$ are the sums of the numbers in column 1, Table I., (with their signs changed): which first factor is

$$1+tx^2+tx^3+\begin{pmatrix} t \\ +t^2 \end{pmatrix}x^4+\dots,$$

and it is as before to be operated on with $[t^{1\dots\infty}]$, viz. the constant term is to be rejected; and further, that there is a second factor $=x+tx^2+t^2x^3+\dots$, the coefficients of which are obtained by summation of the numbers in the several lines of columns 0, 1 of Table I. We have thence column 2 of Table I.

As regards column 3, it shows that the *GF* has a first factor,

$$(1-tx^3)^{-1}(1-tx^4)^{-2}(1-tx^5)^{-4}\dots,$$

where the indices $-1, -2, -4, \dots$ are the sums of the numbers in column 2 of Table I., (with their signs changed): which first factor is

$$=1+tx^3+2tx^4+4tx^5+\begin{pmatrix} 6t \\ +t^2 \end{pmatrix}x^6+\dots,$$

and it is as before to be operated on with $[t^{1\dots\infty}]$, viz. the constant term is to be rejected; and that there is a second factor

$$=x+tx^2+\begin{pmatrix} t \\ +t^2 \end{pmatrix}x^3+\begin{pmatrix} t \\ +t^2 \\ +t^3 \end{pmatrix}x^4+\dots,$$

the coefficients of which are obtained by summation of the numbers in the several lines of columns 0, 1, 2 of Table I.: we have thence column 3 of Table I.

And similarly, by means of columns 0, 1, 2, 3 of Table I., we form the *GF* of column 4; that is, we obtain column 4 of Table I., and so on indefinitely.

To apply the Subsidiary Table to the calculation of the *GF*'s of Table II., the only difference is that the first factors are to be taken without the terms in t^1: thus for Table II. column 3, the first factor of the *GF*

$$= t^2x^6 + 2t^2x^7 + 7t^2x^8 + \binom{14t^2}{+t^3} x^9 + \&\text{c.},$$

the second factor being as for Table I.

$$= x + tx^2 + \binom{t}{+t^2} x^3 + \&\text{c.}$$

The remaining Tables are Tables III. and IV., oxygen root-trees and centre- and bicentre-trees, followed by a Subsidiary Table for the calculation of the *GF*'s: Tables V. and VI., boron root-trees and centre- and bicentre-trees, followed by a Subsidiary Table; and Tables VII. and VIII., carbon root-trees and centre- and bicentre-trees, followed by a Subsidiary Table. The explanations given as to Tables I., II. and the Subsidiary Table apply *mutatis mutandis* to these; and but little further explanation is required: that given in regard to the Subsidiary Table of Tables III. and IV. shows how this limiting case comes under the general method. As to the Subsidiary of Tables V. and VI., it is to be observed that each * line of the Table is calculated from a column of Table V., rejecting the numbers which belong to t^3; thus Table V., column 4, the numbers are

t^1	1	3	5	7	8	9 ...
t^2		1	4	10	21	36 ...
t^3			1	4	11	26 ...;

and taking the sums for the first and second lines only, these are

1, 4, 9, 17, 29, 45,..,

which, taken with a negative sign, are the numbers of the line **GF*, column 5.

And so as to the Subsidiary of Tables VII. and VIII., each * line of the Table is calculated from a column of Table VII., rejecting the numbers which belong to t^4; thus Table VII., column 4, the numbers are

t^1	1	3	8	15	27	43 ...
t^2		1	4	13	33	74 ...
t^3			1	4	14	38 ...
t^4				1	4	14 ...;

and taking the sums for the first, second, and third lines only, these are

1, 4, 13, 32, 74, 155,..,

which, taken with a negative sign, are the numbers of the line **GF*, column 5.

Referring to the foregoing "Edification Diagram," the effect is that we thus introduce the conditions that in a boron-tree the number of component trees $a, b, \ldots$ is at most $(3-1=)\,2$ and that in a carbon-tree the number of component trees $a, b, \ldots$ is at most $(4-1=)\,3$.

TABLE III.—Oxygen Root-Trees.

Index x, or number of knots.	Index t, or number of main branches.	Altitude or number of column.												
		0	1	2	3	4	5	6	7	8	9	10	11	12
1	0	1												
2	1		1											
3	1			1										
	2		1											
4	1				1									
	2			1										
5	1					1								
	2			1	1									
6	1						1							
	2				1	1								
7	1							1						
	2				1	1	1							
8	1								1					
	2					1	1	1						
9	1									1				
	2					1	1	1	1					
10	1										1			
	2						1	1	1	1				
11	1											1		
	2						1	1	1	1	1			
12	1												1	
	2							1	1	1	1	1		
13	1													1
	2							1	1	1	1	1	1	

TABLE IV.—Oxygen Centre- and Bicentre-Trees.

Index x, or number of knots.	Index t, or number of main branches.	Centre-Trees. Altitude or number of column.							Centre.	Total.	Bicentre.	Bicentre-Trees. Altitude.					
		0	1	2	3	4	5	6				0	1	2	3	4	5
1	0	1							1	1	0						
2									0	1	1	1					
3	2		1						1	1	0						
4									0	1	1		1				
5	2			1					1	1	0						
6									0	1	1			1			
7	2				1				1	1	0						
8									0	1	1				1		
9	2					1			1	1	0						
10									0	1	1					1	
11	2						1		1	1	0						
12									0	1	1						1
13	2							1	1	1	0						

Subsidiary Table for *GF*'s of Tables III. and IV.

Index *t*.	Index of *x*. 0	1	2	3	4	5	6	7	8	9	10	11	12	13	
0															*GF*, column 0.
*		−1													*GF*, column 1.
0	1														First factor.
1		1													
2			1												
0		1													Second factor.
*			−1												*GF*, column 2.
0	1														First factor.
1			1												
2					1										
0		1													Second factor.
1			1												
*				−1											*GF*, column 3.
0	1														First factor.
1				1	.	.	.								
2							1								
0		1													Second factor.
1			1	1											
*					−1										*GF*, column 4.
0	1														First factor.
1					1										
2						.	.	.	1						
0		1													Second factor.
1			1	1	1										
*						−1									*GF*, column 5.
0	1														First factor.
1						1									
2											1				
0		1													Second factor.
1			1	1	1	1									

and so on indefinitely; viz. observing that the first factors, as shown by the Table, are $(1-tx)^{-1}[t^{1\cdot 2}]$, $(1-tx^2)^{-1}[t^{1\cdot 2}]$, &c., the Table in fact shows that as regards Table III. the GF's are for

$$\begin{array}{lrl} \text{column} & 0: & x, \\ \text{,,} & 1: & tx\ + t^2x^2\ .\ x, \\ \text{,,} & 2: & tx^2 + t^2x^4\ .\ x + tx^2, \\ \text{,,} & 3: & tx^3 + t^2x^6\ .\ x + t\,(x^2+x^3), \\ \text{,,} & 4: & tx^4 + t^2x^8\ .\ x + t\,(x^2+x^3+x^4), \\ \text{,,} & 5: & tx^5 + t^2x^{10}\,.\ x + t\,(x^2+x^3+x^4+x^5)\,; \end{array}$$

viz. developing as far as t^2, that the successive GF's are

$$\begin{array}{lrl} \text{column} & 0: & x, \\ \text{,,} & 1: & tx^2 + t^2x^3, \\ \text{,,} & 2: & tx^3 + t^2\,(x^4+x^5), \\ \text{,,} & 3: & tx^4 + t^2\,(x^5+x^6+x^7), \\ \text{,,} & 4: & tx^5 + t^2\,(x^6+x^7+x^8+x^9), \\ \text{,,} & 5: & tx^6 + t^2\,(x^7+x^8+x^9+x^{10}+x^{11})\,; \end{array}$$

&c., agreeing with Table III.

And so also it shows that, as regards Table IV. (centre part), the GF's of the successive columns are for

$$\begin{array}{lrl} \text{column} & 0: & x, \\ \text{,,} & 1: & t^2x^2\ .\ x, \\ \text{,,} & 2: & t^2x^4\ .\ x, \\ \text{,,} & 3: & t^2x^6\ .\ x, \\ \text{,,} & 4: & t^2x^8\ .\ x, \\ \text{,,} & 5: & t^2x^{10}\,.\ x\,; \\ & \vdots & \end{array}$$

viz. that the successive GF's are x, t^2x^3, t^2x^5, t^2x^7, t^2x^9, t^2x^{11}, .., agreeing in fact with Table IV.

Table V.—Boron Root-trees.

Index x, or number of knots.	Index t, or number of main branches.	Altitude or number of column.													
		0	1	2	3	4	5	6	7	8	9	10	11	12	
1	0	1	1												
	Total	1	**1**												
2	1		1	1											
	Total		1	**1**											
3	1			1	1										
	2		1		1										
	Total		1	1	**2**										
4	1			1	1	2									
	2			1		1									
	3		1			1									
	Total		1	2	1	**4**									
5	1				2	1	3								
	2			2	1		3								
	3			1			1								
	Total			3	3	1	**7**								
6	1				2	3	1	6							
	2			1	3	1		5							
	3			2	1			3							
	Total			3	6	4	1	**14**							
7	1				1	5	4	1	11						
	2			1	6	4	1		12						
	3			2	3	1			6						
	Total			3	10	10	5	1	**29**						
8	1				1	7	9	5	1	23					
	2				7	10	5	1		23					
	3			2	7	4	1			14					
	Total			2	15	21	15	6	1	**60**					
9	1					8	17	14	6	1	46				
	2				9	21	15	6	1		52				
	3			1	11	11	5	1			29				
	Total			1	20	40	37	21	7	1	**127**				
10	1					9	29	32	20	7	1	98			
	2				7	36	37	21	7	1		109			
	3			1	18	26	16	6	1			68			
	Total			1	25	71	82	59	28	8	1	**275**			
11	1					7	45	66	53	27	8	1	207		
	2				7	59	82	59	28	8	1		244		
	3				21	53	43	22	7	1			147		
	Total				28	119	170	147	88	36	9	1	**598**		
12	1					7	66	127	125	81	35	9	1	451	
	2				4	82	165	147	88	36	9	1		532	
	3				26	102	105	66	29	8	1			337	
	Total				30	191	336	340	242	125	45	10	1	**1320**	
13	1					4	89	231	274	213	117	44	10	1	983
	2				3	114	316	340	242	125	45	10	1		1196
	3				26	175	236	177	96	37	9	1			757
	Total				29	293	641	748	612	375	171	55	11	1	**2936**

TABLE VI.—Boron Centre- and Bicentre-Trees.

Index x, or number of knots.	Index t, or number of main branches.	Centre-Trees. Altitude or number of column.								Centre.	Grand Total.	Bicentre.	Bicentre-Trees. Altitude.					
		0	1	2	3	4	5	6					0	1	2	3	4	5
1	0	1	1															
	Total	1	1							1	1	0						
2																		
										0	1	1	1					
3	2		1	1														
	Total		1	1						1	1	0						
4	2		1	1														
	Total		1	1						1	2	1		1				
5	2			1	1													
	Total			1	1					1	2	1		1				
6	2			1	1													
	3			1	1													
	Total			2	2					2	4	2		1	1			
7	2			1	1	2												
	3			2		2												
	Total			3	1	4				4	6	2			2			
8	2				2	2												
	3			2	1	3												
	Total			2	3	5				5	11	6			5	1		
9	2				5	1	6											
	3			1	3		4											
	Total			1	8	1	10			10	18	8			5	3		
10	2				5	3	8											
	3			1	9	1	11											
	Total			1	14	4	19			19	37	18			6	11	1	
11	2				6	11	1	18										
	3				14	4		18										
	Total				20	15	1	36		36	66	30			4	22	4	
12	2				4	22	4	30										
	3				21	16	1	38										
	Total				25	38	5	68		68	135	67			3	44	19	1
13	2				3	44	19	1	67									
	3				24	42	5		71									
	Total				27	86	24	1	138	138	265	127			1	68	53	5

Subsidiary Table for *GF*'s of Tables V. and VI.

Index *t*.	Index of *x*.														
	0	1	2	3	4	5	6	7	8	9	10	11	12	13	
0		1													*GF*, column 0.
*		−1													*GF*, column 1.
0	1														First factor.
1		1													
2			1												
3				1											
0		1													Second factor.
*			−1	−1											*GF*, column 2.
0	1														First factor.
1			1	1											
2					1	1	1								
3							1	1	1	1					
0		1													Second factor.
1			1												
2				1											
*				−1	−2	−2	−1	−1							*GF*, column 3.
0	1														First factor.
1				1	2	2	1	1							
2							1	2	5	5	6	4	3		
3										1	2	5	9		
0		1													Second factor.
1			1	1	1										
2				1	1	2	1	1							
*					−1	−3	−5	−7	−8	−9	−7	−7	−4	−3	*GF*, column 4.
0	1														First factor.
1					1	3	5	7	8	9	7	7	4		
2									1	3	11	22	44		
3													1		
0		1													Second factor.
1			1	1	2	2	2	1	1						
2				1	1	3	4	7	7	7	7	7	4	3	
*						−1	−4	−9	−17	−29	−45	−66	−89	−118	*GF*, column 5.
0	1														First factor.
1						1	4	9	17	29	45	66	89		
2											1	4	19		
0		1													Second factor.
1			1	1	2	3	5	6	8	8	9	7	7	4	
2				1	1	3	5	11	17	30	43	66	86	117	
*							−1	−5	−14	−32	−66	−127	−231	−405	*GF*, column 6.
0	1														First factor.
1							1	5	14	32	66	127	231		
2													1		
0		1													Second factor.
1			1	1	2	3	6	10	17	25	38	52	73	93	
2				1	1	3	5	12	22	45	80	148	251	433	

Subsidiary Table for *GF*'s of Tables V. and VI. (*continued*).

Index *t*.	Index of *x*.														
	0	1	2	3	4	5	6	7	8	9	10	11	12	13	
*								-1	-6	-20	-53	-125	-274	-571	*GF*, column 7.
0	1														First factor.
1								1	6	20	53	125	274		
0		1													Second factor.
1			1	1	2	3	6	11	22	39	70	118	200	324	
2				1	1	3	5	12	23	51	101	207	398	773	
*									-1	-7	-27	-81	-213	-516	*GF*, column 8.
0	1														First factor.
1									1	7	27	81	213		
0		1													Second factor.
1			1	1	2	3	6	11	23	45	90	171	325	598	
2				1	1	3	5	12	23	52	108	235	486	1015	
*										-1	-8	-35	-117	-338	*GF*, column 9.
0	1														First factor.
1										1	8	35	117		
0		1													Second factor.
1			1	1	2	3	6	11	23	46	97	198	406	811	
2				1	1	3	5	12	23	52	109	243	522	1140	
*											-1	-9	-44	-162	*GF*, column 10.
0	1														First factor.
1											1	9	44		
0		1													Second factor.
1			1	1	2	3	6	11	23	46	98	206	441	928	
2				1	1	3	5	12	23	52	109	244	531	1185	
*												-1	-10	-54	*GF*, column 11.
0	1														First factor.
1												1	10		
0		1													Second factor.
1			1	1	2	3	6	11	23	46	98	207	450	972	
2				1	1	3	5	12	23	52	109	244	532	1195	
*													-1	-11	*GF*, column 12.
0	1														First factor.
1													1		
0		1													Second factor.
1			1	1	2	3	6	11	23	46	98	207	451	982	
2				1	1	3	5	12	23	52	109	244	532	1196	
*														-12	*GF*, column 13.
0	1														First factor.
0		1													Second factor.
1			1	1	2	3	6	11	23	46	98	207	451	983	
2				1	1	3	5	12	23	52	109	244	532	1196	

TABLE VII.—Carbon Root-trees.

Index x, or number of knots.	Index t, or number of main branches.	Altitude or number of column.												
		0	1	2	3	4	5	6	7	8	9	10	11	12
1	0	1	1											
	Total	1	1											
2	1		1	1										
	Total		1	1										
3	1			1	1									
	2		1		1									
	Total		1	1	2									
4	1			1	1	2								
	2			1		1								
	3		1			1								
	Total		1	2	1	4								
5	1			1	2	1	4							
	2			2	1		3							
	3			1			1							
	4		1				1							
	Total		1	4	3	1	9							
6	1				4	3	1	8						
	2			2	3	1		6						
	3			2	1			3						
	4			1				1						
	Total			5	8	4	1	18						
7	1				4	8	4	1	17					
	2			2	8	4	1		15					
	3			3	3	1			7					
	4			2	1				3					
	Total			7	16	13	5	1	42					

TABLE VII. (*continued*).

Index x, or number of knots.	Index t, or number of main branches.	Altitude or number of column.													
		0	1	2	3	4	5	6	7	8	9	10	11	12	
8	1				5	15	13	5	1	39					
	2			1	13	13	5	1		33					
	3			3	9	4	1			17					
	4			3	3	1				7					
	Total			7	30	33	19	6	1	**96**					
9	1				4	27	32	19	6	1	89				
	2			1	22	33	19	6	1		82				
	3			3	17	14	5	1			40				
	4			4	9	4	1				18				
	Total			8	52	78	57	26	7	1	**229**				
10	1				4	43	74	56	26	7	1	211			
	2				29	74	57	26	7	1		194			
	3			3	34	38	20	6	1			102			
	4			4	18	14	5	1				42			
	Total			7	85	169	156	89	34	8	1	**549**			
11	1				3	67	155	151	88	34	8	1	507		
	2				40	154	156	89	34	8	1		482		
	3			2	54	95	63	27	7	1			249		
	4			5	38	39	20	5	1				108		
	Total			7	135	355	394	272	130	43	9	1	**1346**		
12	1				2	97	316	374	267	129	43	9	1	1238	
	2				46	297	389	273	130	43	9	1		1188	
	3			1	88	218	184	96	35	8	1			631	
	4			4	66	100	64	27	7	1				269	
	Total			5	202	712	953	770	439	181	53	10	1	**3326**	
13	1				1	136	612	889	743	432	180	52	10	1	3056
	2				55	550	929	770	439	181	53	10	1		2988
	3			1	127	474	491	309	138	44	9	1			1594
	4			4	117	239	190	97	35	8	1				691
	Total			5	300	1399	2222	2065	1355	665	243	63	11	1	**8329**

TABLE VIII.—Carbon Centre- and Bicentre-Trees.

Index *x*, or number of knots.	Index *t*, or number of main branches.	Centre-Trees. Altitude or number of column.								Centre.	Grand Total.	Bicentre.	Bicentre-Trees. Altitude.					
		0	1	2	3	4	5	6					0	1	2	3	4	5
1	0	1	1															
	Total	1	1							1	1	0						
2																		
										0	1	1	1					
3	2		1	1														
	Total		1	1						1	1	0						
4	2		1	1														
	Total		1	1						1	2	1		1				
5	2		1	1	2													
	Total		1	1	2					2	3	1		1				
6	2			1	1													
	3			1	1													
	Total			2	2					2	5	3		2	1			
7	2			2	1	3												
	3			2		2												
	4			1		1												
	Total			5	1	6				6	9	3		1	2			
8	2			1	2	3												
	3			3	1	4												
	4			2		2												
	Total			6	3	9				9	18	9		1	7	1		
9	2			1	7	1	9											
	3			3	3		6											
	4			4	1		5											
	Total			8	11	1	20			20	35	15			12	3		
10	2				12	3	15											
	3			3	11	1	15											
	4			4	3		7											
	Total			7	26	4	37			37	75	38			23	14	1	
11	2				23	14	1	38										
	3			2	24	4		30										
	4			5	12	1		18										
	Total			7	59	19	1	86		86	159	73			30	39	4	
12	2				30	39	4	73										
	3			1	54	19	1	75										
	4			4	27	4		35										
	Total			5	111	62	5	183		183	357	174			42	108	23	1
13	2				42	108	23	1	174									
	3			1	88	63	5		157									
	4			4	63	20	1		88									
	Total			5	193	191	29	1	419	419	799	380			47	244	84	5

Subsidiary Table for *GF*'s of Tables VII. and VIII.

Index *t*.	Index of *x*. 0	1	2	3	4	5	6	7	8	9	10	11	12	13	
0		1													*GF*, column 0.
*		−1													*GF*, column 1.
0	1														First factor.
1		1													
2			1												
3				1											
4					1										
0		1													Second factor.
*			−1	−1	−1										*GF*, column 2.
0	(1)														First factor.
1			1	1	1										
2					1	1	2	1	1						
3							1	1	2	2	2	1	1		
4									1	1	2	2	3		
0		1													Second factor.
1			1												
2				1											
3					1										
*				−1	−2	−4	−4	−5	−4	−4	−3	−2	−1	−1	*GF*, column 3.
0	(1)														First factor.
1				1	2	4	4	5	4	4	3	2	1		
2							1	2	4	12	23	30	42		
3										1	2	7	16		
4													1		
0		1													Second factor.
1			1	1	1	1									
2				1	1	2	2	2	1	1					
3					1	1	2	3	3	3	3	2	1	1	
*					−1	−3	−8	−15	−27	−43	−67	−97	−136	−183	*GF*, column 4.
0	(1)														First factor.
1					1	3	8	15	27	43	67	97	136		
2									1	3	14	39	108		
3													1		
0		1													Second factor.
1			1	1	2	3	4	4	5	4	4	3	2	1	
2				1	1	3	5	10	14	23	29	40	46	55	
3					1	1	3	6	12	20	37	56	89	128	
*						−1	−4	−13	−32	−74	−155	−316	−612	−1160	*GF*, column 5.
0	(1)														First factor.
1						1	4	13	32	74	155	316	612		
2											1	4	23		
0		1													Second factor.
1			1	1	2	4	7	12	20	31	47	70	99	137	
2				1	1	3	6	14	27	56	103	194	343	605	
3					1	1	3	7	16	34	75	151	307	602	
*							−1	−5	−19	−56	−151	−374	−889	−2032	*GF*, column 6.
0	1														First factor.
1							1	5	19	56	151	374	889		
2													1		
0		1													Second factor.
1			1	1	2	4	8	16	33	63	121	225	415	749	
2				1	1	3	6	15	32	75	160	350	732	1534	
3					1	1	3	7	17	39	95	214	491	1093	

Subsidiary Table for *GF*'s of Tables VII. and VIII. (*continued*).

Index *t*.	Index of *x*. 0	1	2	3	4	5	6	7	8	9	10	11	12	13	
*								– 1	– 6	– 26	– 88	– 267	– 743	– 1968	*GF*, column 7.
0	(1)														First factor.
1								1	6	26	88	267	743		
0		1													Second factor.
1			1	1	2	4	8	17	38	82	177	376	789	1638	
2				1	1	3	6	15	33	81	186	439	1005	2304	
3					1	1	3	7	17	40	101	241	587	1402	
*									– 1	– 7	– 34	– 129	– 432	– 1320	*GF*, column 8.
0	(1)														First factor.
1									1	7	34	129	432		
0		1													Second factor.
1			1	1	2	4	8	17	39	88	203	464	1056	2381	
2				1	1	3	6	15	33	82	193	473	1135	2743	
3					1	1	3	7	17	40	102	248	622	1540	
*										– 1	– 8	– 43	– 180	– 657	*GF*, column 9.
0	(1)														First factor.
1										1	8	43	180		
0		1													Second factor.
1			1	1	2	4	8	17	39	89	210	498	1185	2813	
2				1	1	3	6	15	33	82	194	481	1178	2924	
3					1	1	3	7	17	40	102	249	630	1584	
*											– 1	– 9	– 53	– 242	*GF*, column 10.
0	(1)														First factor.
1											1	9	53		
0		1													Second factor.
1			1	1	2	4	8	17	39	89	211	506	1228	2993	
2				1	1	3	6	15	33	82	194	482	1187	2977	
3					1	1	3	7	17	40	102	249	631	1593	
*												– 1	– 10	– 63	*GF*, column 11.
0	(1)														First factor.
1												1	10	63	
0		1													Second factor.
1			1	1	2	4	8	17	39	89	211	507	1237	3048	
2				1	1	3	6	15	33	82	194	482	1188	2987	
3					1	1	3	7	17	40	102	249	631	1594	
*													– 1	– 11	*GF*, column 12.
0	(1)														First factor.
1													1	11	
0		1													Second factor.
1			1	1	2	4	8	17	39	89	211	507	1238	3055	
2				1	1	3	6	15	33	82	194	482	1188	2988	
3					1	1	3	7	17	40	102	249	631	1594	
*														– 1	*GF*, column 13.
0	(1)														First factor.
1														1	
0		1													Second factor.
1			1	1	2	4	8	17	39	89	211	507	1238	3056	
2				1	1	3	6	15	33	82	194	482	1188	2988	
3					1	1	3	7	17	40	102	249	634	1594	

I annex the following two Tables of (centre- and bicentre-) trees as far as I have completed them.

TABLE A.

Knots.	Valency not greater than									Gen.
	0	1	2 Oxygen.	3 Boron.	4 Carbon.	5	6	7	8	
1	1	1	1	1	1	1	1	1	1	1
2		1	1	1	1	1	1	1	1	1
3			1	1	1	1	1	1	1	1
4			1	2	2	2	2	2	2	2
5			1	2	3	3	3	3	3	3
6			1	4	5	6	6	6	6	6
7			1	6	9	10	11	11	11	11
8			1	11	18	21	22	23	23	23
9			1	18	35	42	45	46	47	47
10			1	37	75					106
11			1	66	159					235
12			1	135	357					551
13			1	265	799					1301

TABLE B.

Knots.	Actual Valency.								
	0	1	2	3	4	5	6	7	8
1	1								
2		1							
3			1						
4			1	1					
5			1	1	1				
6			1	3	1	1			
7			1	5	3	1	1		
8			1	10	7	3	1	1	
9			1	17	17	7	3	1	1
10			1	36	38				
11			1	65	93				
12			1	134	222				
13			1	264	534				

In A, the columns 2, 3, 4, and the last column are the totals given by the Tables IV., VI., VIII., and II., and the remaining numbers of columns 5, 6, 7, 8 have been found by trial; and, in B, the several columns are the differences of the columns of A. The signification is obvious; for instance, if the number of knots is $=9$, then Table A, if the valency, or the maximum number of branches from a knot,

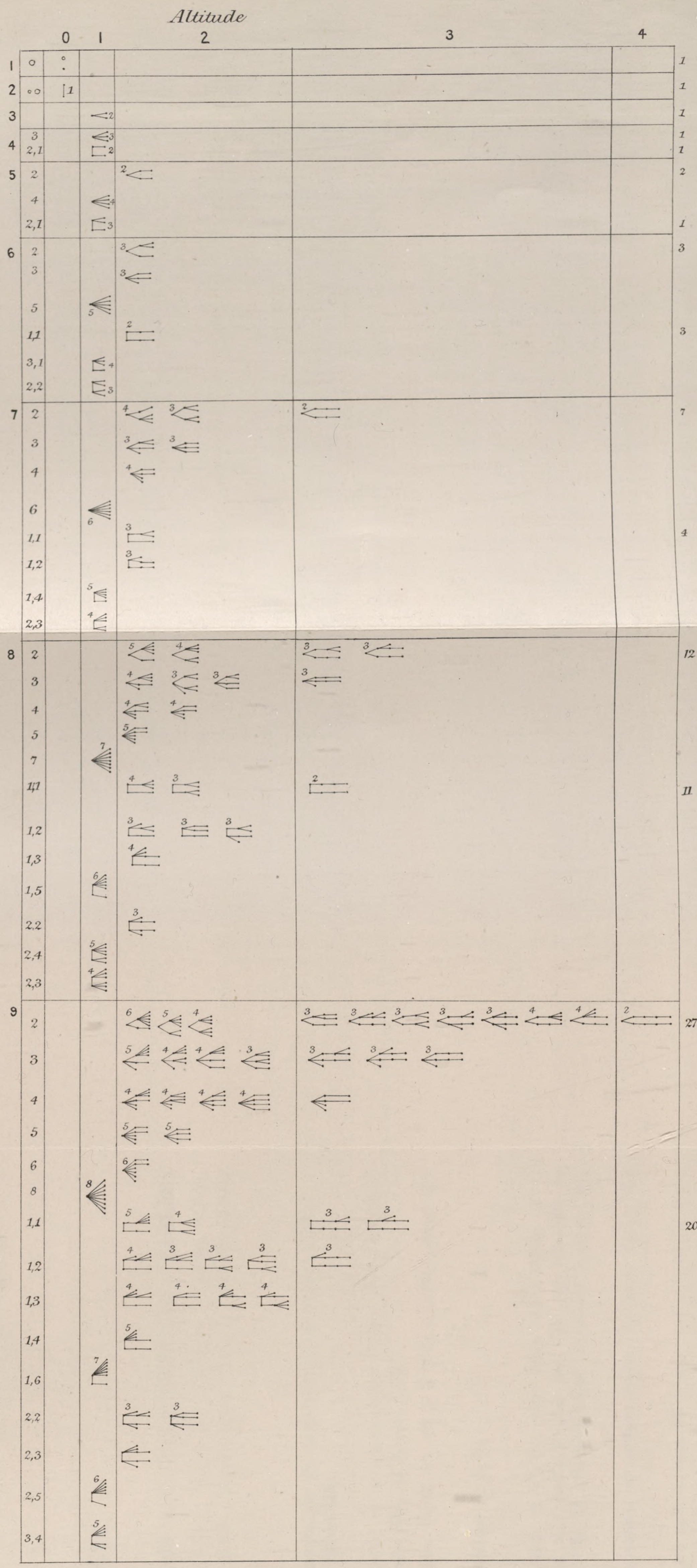
is =	2,	3,	4,	5,	6,	7,	8 or any greater number,
No. of trees =	1,	18,	35,	42,	45,	46,	47:

viz. with 9 knots the tree can have at most 8 branches from a knot, so that the number of trees having at most 8 branches from a knot is $=47$, the whole number of trees with 9 knots; and so the number of knots being as before $=9$, Table B shows that the number of 47 is made up of the numbers

1, 17, 17, 7, 3, 1, 1;

viz. 1	is the No. of trees,	at most	2	branches from	a knot,				
17	„	„	3	„	„	at least one	3-branch knot.		
17	„	„	4	„	„	„	4	„	
7	„	„	5	„	„	„	5	„	
3	„	„	6	„	„	„	6	„	
1	„	„	7	„	„	„	7	„	
1	„	„	8	„	„	„	8	„	

I annex also a plate showing the figures of the $1+1+2+3+6+11+23+47$ trees of 1, 2, 3, .., 9 knots, classified according to their altitudes and number of main branches; and as to the bicentre-trees, according to the number of main branches from *each* point of the bicentre. The affixed numbers show in each case the greatest number of branches from a knot; so that when this is (2), the knots may be oxygen-, boron-, carbon-, &c., atoms; when (3), boron-, carbon-, &c., atoms; when (4), carbon-, &c., atoms; and so on.



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