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A GEOMETRICAL ILLUSTRATION OF THE CUBIC TRANSFORMATION IN ELLIPTIC FUNCTIONS.

[From the *Quarterly Journal of Pure and Applied Mathematics*, vol. XIII. (1875), pp. 211—216.]

CONSIDER the cubic curve

$$x^3 + y^3 + z^3 + 6lxyz = 0.$$

If through one of the inflexions $z = 0$, $x + y = 0$, we draw an arbitrary line $z = u(x + y)$, we have at the other intersections of this line with the curve

$$u\{u^2(x+y)^2 + 6lxy\} + x^2 - xy + y^2 = 0;$$

that is,

$$(u^3 + 1)(x^2 + y^2) + 2xy(u^3 + 3lu - \tfrac{1}{2}) = 0;$$

and from this equation it appears that the ratio $x : y$ is given as a function involving the square root of

$$(u^3 + 3lu - \tfrac{1}{2})^2 - (u^3 + 1)^2,$$

which, rejecting a factor 3, is

$$= (2u^3 + 3lu + \tfrac{1}{2})(lu - \tfrac{1}{2}).$$

It may be noticed that $lu - \frac{1}{2} = 0$ gives the value of u, which in the equation $z = u(x + y)$ belongs to the tangent at the inflexion; and $2u^3 + 3lu + \frac{1}{2} = 0$ gives the values which belong to the three tangents from the inflexion.

It thus appears that the coordinates x, y, z of any point of the curve can be expressed as proportional to functions of u involving the radical

$$\sqrt{\{(lu - \tfrac{1}{2})(2u^3 + 3lu + \tfrac{1}{2})\}},$$

and the theory of the curve is connected with that of a quasi-elliptic integral depending on this radical.

Taking ω an imaginary cube root of unity, write

$$\omega x + \omega^2 y - 2lz = x',$$

$$\omega^2 x + \omega y - 2lz = y',$$

$$x + y - 2lz = z';$$

then we have

$$x'y'z' = x^3 + y^3 - 8l^3z^3 + 6lxyz = x^3 + y^3 + z^3 + 6lxyz - (1 + 8l^3)\, z^3.$$

Also

$$-6lz = x' + y' + z', \quad z^3 = \frac{-1}{216l^3}(x' + y' + z')^3,$$

whence

$$(x' + y' + z')^3 - \frac{216l^3}{1+8l^3}\, x'y'z' = \frac{216l^3}{1+8l^3}(x^3 + y^3 + z^3 + 6lxyz);$$

so that, putting

$$m^3 = \frac{-l^3}{1+8l^3},$$

or, what is the same thing,

$$8l^3m^3 + l^3 + m^3 = 0,$$

the equation of the curve is

$$(x' + y' + z')^3 + 216m^3x'y'z' = 0;$$

and if we write

$$x' : y' : z' = X^3 : Y^3 : Z^3,$$

then the original curve is transformed into

$$(X^3 + Y^3 + Z^3)^3 + 216m^3X^3Y^3Z^3 = 0,$$

a curve of the ninth order breaking up into three cubic curves, one of which is

$$X^3 + Y^3 + Z^3 + 6mXYZ = 0,$$

and for the other two we write herein $m\omega$ and $m\omega^2$ respectively in place of m. Attending only to the first curve, we have

$$x^3 + y^3 + z^3 + 6lxyz = 0,$$

$$X^3 + Y^3 + Z^3 + 6mXYZ = 0,$$

as corresponding curves, the corresponding points being connected by the relation

$$\omega x + \omega^2 y - 2lz : \omega^2 x + \omega y - 2lz : x + y - 2lz = X^3 : Y^3 : Z^3,$$

or, for convenience, we may write

$$\begin{aligned} \omega x + \omega^2 y - 2lz &= X^3, \text{ giving} & 3x &= \omega^2 X^3 + \omega Y^3 + Z^3, \\ \omega^2 x + \omega y - 2lz &= Y^3, & 3y &= \omega X^3 + \omega^2 Y^3 + Z^3, \\ x + y - 2lz &= Z^3, & -6lz &= X^3 + Y^3 + Z^3. \end{aligned}$$

This is a (1, 3) correspondence; viz. to a given point on the curve (m), there corresponds one point on (l); but to a given point on (l), three points on (m). As to the first case, this is obvious. As to the second case, if the point (x, y, z) is given, then the corresponding point (X, Y, Z) on the other curve will lie on one of the three lines

$$Y^3(\omega x+\omega^2 y-2lz)-X^3(\omega^2 x+\omega y-2lz)=0\,;$$

each of these intersects the curve (m) in three points: but of the points in the same line it is only one which is a corresponding point of (x, y, z), and the number of the corresponding points is consequently the same as the number of lines, viz. it is $=3$.

We infer that the above equations lead to a cubic transformation of the quasi-elliptic integral

$$\int du\div\sqrt{\{(lu-\tfrac{1}{2})(2u^3+3lu+\tfrac{1}{2})\}},$$

into one of the like form

$$\int dv\div\sqrt{\{(mv-\tfrac{1}{2})(2v^3+3mv+\tfrac{1}{2})\}}\,;$$

and this is now to be verified.

We have, as before, the line $z=u(x+y)$ meeting the curve (l) in the points

$$(u^3+1)(x^2+y^2)+2xy(u^3+3lu-\tfrac{1}{2})=0\,;$$

and if similarly through an inflexion of the curve (m) we take the line $Z=v(X+Y)$, this meets the curve in the points

$$(v^3+1)(X^2+Y^2)+2XY(v^3+3mv-\tfrac{1}{2})=0.$$

Then if (x, y, z), (X, Y, Z) are taken to be the corresponding points as above, we can obtain v as a function of u. We, in fact, have

$$\begin{aligned}-2lu=\frac{-2lz}{x+y}=\frac{X^3+Y^3+Z^3}{-X^3-Y^3+2Z^3}&=\frac{X^3+Y^3+v^3(X+Y)^3}{-(X^3+Y^3)+2v^3(X+Y)^3}\\&=\frac{X^2-XY+Y^2+v^3(X+Y)^2}{-X^2+XY-Y^2+2v^3(X+Y)^2},\\&=\frac{(v^3+1)(X^2+Y^2)+(2v^3-1)XY}{(2v^3-1)(X^2+Y^2)+(4v^3+1)XY};\end{aligned}$$

or, since we have

$$(v^3+1)(X^2+Y^2)+2XY(v^3+3mu-\tfrac{1}{2})=0,$$

that is,

$$X^2+Y^2\,:\,XY=-2v^3-6mv+1\,:\,v^3+1,$$

the equation becomes

$$\begin{aligned}-2lu&=\frac{-6mv(v^3+1)}{(2v^3-1)(-2v^3-6mv+1)+(4v^3+1)(v^3+1)}\\&=\frac{-6mv(v^3+1)}{-3v(4mv^3-3v^2-2m)};\end{aligned}$$

or say,

$$-lu = m(v^3+1)(\div), \text{ where the denominator } = 4mv^3 - 3v^2 - 2m.$$

This may also be written

$$-(lu-\tfrac{1}{2}) = 3v^2(mv-\tfrac{1}{2}) \div .$$

Proceeding to calculate $2u^3+3lu+\frac{1}{2}$, omitting the denominator $(4mv^3-3v^2-2m)^3$, this is

$$-\frac{2m^3}{l^3}(v^3+1)^3 - 3m(v^3+1)(4mv^3-3v^2-2m)^2 + \tfrac{1}{2}(4mv^3-3v^2-2m)^3;$$

or, observing that

$$m^3 = \frac{-l^3}{1+8l^3},$$

that is,

$$l^3 = \frac{-m^3}{1+8m^3} \text{ or } -\frac{m^3}{l^3} = 1+8m^3,$$

the numerator is

$$= 2(1+8m^3)(v^3+1)^3 - 3m(v^3+1)(4mv^3-3v^2-2m)^2 + \tfrac{1}{2}(4mv^3-3v^2-2m)^3,$$

which is found to be identically

$$= (2v^3+3mv+\tfrac{1}{2})(v^3+6mv-2)^2;$$

viz. we have

$$2u^3+3lu+\tfrac{1}{2} = (2v^3+3mv+\tfrac{1}{2})(v^3+6mv-2)^2 \div (4mv^3-3v^2-2m)^3,$$

and hence

$$(lu-\tfrac{1}{2})(2u^3+3lu+\tfrac{1}{2}) = -3(mv-\tfrac{1}{2})(2v^3+3mv+\tfrac{1}{2})(v^3+6mv-2)^2 v^2 \div (4mv^3-3v^2-2m)^4.$$

Moreover, we find

$$ldu = 3mdv \,.\, v(v^3+6mv-2) \div (4mv^3-3v^2-2m)^2,$$

and we thence have

$$\frac{ldu}{\sqrt{\{(lu-\frac{1}{2})(2u^3+3lu+\frac{1}{2})\}}} = \sqrt{(-3)}\frac{mdv}{\sqrt{\{(mv-\frac{1}{2})(2v^3+3mv+\frac{1}{2})\}}};$$

viz. this differential equation corresponds to the integral equation

$$-lu = m(v^3+1) \div (4mv^3-3v^2-2m),$$

where $8l^3m^3+l^3+m^3=0$, which corresponds to the modular equation.

It may be remarked that, if v is the same function of u', l, m that u is of v, m, l; viz. if

$$-mv = l(u'^3+1) \div (4lu'^3-3u'^2-2m'),$$

then

$$\frac{mdv}{\sqrt{\{(mv-\frac{1}{2})(2v^3+3mv+\frac{1}{2})\}}} = \sqrt{(-3)}\frac{-ldu'}{\sqrt{\{(lu'-\frac{1}{2})(2u'^3+3lu'+\frac{1}{2})\}}},$$

and consequently

$$\frac{du}{\sqrt{\{(lu-\frac{1}{2})(2u^3+3lu+\frac{1}{2})\}}} = \frac{-3du'}{\sqrt{\{(lu'-\frac{1}{2})(2u'^3+3lu'+\frac{1}{2})\}}},$$

which accords with the general theory of the cubic transformation.

We may inquire into the relation between the absolute invariants of the two curves. Taking the absolute invariant to be

$$\Omega = \frac{64S^3 - T^2}{64S^3},$$

where S and T bear the usual significations, we have for the one curve

$$\Omega = \frac{(1+8l^3)^3}{64l^3(1-l^3)^3},$$

and for the other curve

$$\Omega' = \frac{(1+8m^3)^3}{64m^3(1-m^3)^3},$$

and, as above, $8l^3m^3 + l^3 + m^3 = 0$: writing herein

$$l^3 = -\frac{1}{8\alpha'},\ m^3 = -\frac{1}{8\beta'},$$

the relation between α', β' is simply $\alpha' + \beta' = 1$; and the values of Ω, Ω' are found to be

$$\Omega = \frac{64\alpha'(1-\alpha')^3}{(1+8\alpha')^3},\ \Omega' = \frac{64\beta'(1-\beta')^3}{(1+8\beta')^3};$$

viz. the required relation is given by the elimination of α', β' from these three equations. Or, what is the same thing, writing $\alpha' = \frac{1}{2} + \theta$, and therefore $\beta' = \frac{1}{2} - \theta$, we have

$$(5+8\theta)^3\,\Omega\ = 4(1+2\theta)\,(1-2\theta)^3,$$
$$(5-8\theta)^3\,\Omega' = 4(1+2\theta)^3(1-2\theta),$$

and the elimination of θ from these equations gives the required relation between Ω and Ω'.

It of course follows that, if we have a *cubic* transformation

$$\frac{dx}{\sqrt{\{(a,\ b,\ c,\ d,\ e \between x,\ 1)\}^4}} = \frac{Cdx'}{\sqrt{\{(a',\ b',\ c',\ d',\ e' \between x',\ 1)\}^4}},$$

then the absolute invariants Ω, Ω' of the two quartic functions are connected by the above relation. I have obtained this result, by reducing the radicals to the standard forms

$$\sqrt{(1-x^2\,.\,1-k^2x^2)},\ \sqrt{(1-x'^2\,.\,1-\lambda^2x'^2)},$$

from the known modular equation as represented by the equations

$$\lambda^2 = \frac{\alpha^3(2+\alpha)}{1+2\alpha},\ k^2 = \frac{\alpha(2+\alpha)^3}{(1+2\alpha)^3};$$

viz. the values of the absolute invariants

$$\left(=1-\frac{27J^2}{I^3},\ 1-\frac{27J'^2}{I'^3}\right),$$

are

$$\Omega = \frac{108k^2(1-k^2)^4}{(k^4+14k^2+1)^3},\ \Omega' = \frac{108\lambda^2(1-\lambda^2)^4}{(\lambda^4+14\lambda^2+1)^3},$$

but the method of effecting this is by no means obvious.