617.

ON THE SCALENE TRANSFORMATION OF A PLANE CURVE.

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THE transformation by reciprocal radius vectors can be effected mechanically by Sylvester's Peaucellier-cell. But, employing a more general cell (considered incidentally by him) which may be called the scalene-cell, we have the scalene transformation in question*; viz. if, in two curves, r, r' are radius vectors belonging to the same angle (or say opposite angles) θ, then the relation between r, r' is

$$rr'(r+r')+(m^2-l^2)\,r+(m^2-n^2)\,r'=0\,;$$

or, as this may also be written,

$$r^2+\left(r'-\frac{l^2-m^2}{r'}\right)r+m^2-n^2=0.$$

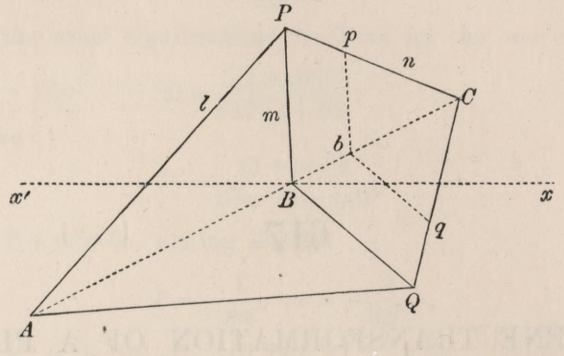
The transformation is, it will be seen, an interesting one for its own sake, independently of the remarkably simple mechanical construction, viz. the scalene cell is simply a system of 3 pairs of equal rods PA, QA; PB, QB; PC, QC (fig. 1, p. 528), jointed together at and capable of rotating about the points P, Q, A, B, C; the three lengths PA, PB, PC (say these are $=l$, m, n) are all of them unequal: in the case of any two of them equal, we have Peaucellier or isosceles cell. The effect of the arrangement is that the points A, B, C are retained in a right line, the distances BA, $=r'$, and BC, $=r$, being connected by the above-mentioned equation; so that taking B as a fixed point, if the point A describe any given curve, the point C will describe the corresponding or transformed curve.

In the case where the given curve is a right line or a circle, we may through B draw at right angles to the curve the axis $x'Bx$: viz. in the case of the circle,

* The transformation itself, and doubtless many of the results obtained by means of it, are familiar to Prof. Sylvester; and I abandon all claim to priority.

the axis $x'Bx$ passes through its centre; and we measure the angle θ from this line, viz. we write $\angle xBC = \angle x'BA = \theta$.

Fig. 1.



Suppose, first, that the locus of A is a right line, or a circle passing through B. Its equation is $r' = \frac{c}{\cos\theta}$ or $= c\cos\theta$; and we accordingly have for the transformed curve

$$r^2 + \left(\frac{c}{\cos\theta} - \frac{l^2 - m^2}{c\cos\theta}\right) r + m^2 - n^2 = 0,$$

or else

$$r^2 + \left(c\cos\theta - \frac{l^2 - m^2}{c\cos\theta}\right) r + m^2 - n^2 = 0;$$

viz. multiplying in each case by $r\cos\theta$, and then writing $r\cos\theta = x$, $r^2 = x^2 + y^2$, the equations become

$$x(x^2 + y^2) + c(x^2 + y^2) - \frac{l^2 - m^2}{c} x^2 \qquad + (m^2 - n^2)x = 0,$$

and

$$x(x^2 + y^2) + cx^2 \qquad - \frac{l^2 - m^2}{c}(x^2 + y^2) + (m^2 - n^2)x = 0;$$

viz. in each case the curve is a circular cubic passing through the origin B and having an asymptote parallel to the axis of y. The curve is nodal, if $m = n$, viz. in this case the origin is a node: or if $c = \sqrt{(l^2 - n^2)} + \sqrt{(m^2 - n^2)}$.

Suppose next that the locus of A is a circle, centre at a distance $= \gamma$ along Bx' and radius $= h$: we have

$$r'^2 - 2\gamma r'\cos\theta + \gamma^2 - h^2 = 0,$$

viz. if

$$\gamma^2 - h^2 = -(l^2 - m^2),$$

or, what is the same thing,

$$h^2 + m^2 = \gamma^2 + l^2,$$

then we have

$$r' - \frac{l^2 - m^2}{r'} = 2\gamma\cos\theta,$$

and the transformed curve is

$$r^2 + 2\gamma r\cos\theta + m^2 - n^2 = 0,$$

or, as this may be written,

$$r^2 + 2\gamma r\cos\theta + \gamma^2 - f^2 = 0,$$

where $\gamma^2 - f^2 = m^2 - n^2$, that is, $f^2 + m^2 = \gamma^2 + n^2$; viz. this is a concentric circle radius f.

The theorem may be presented as follows. Consider two concentric circles, centre O and radii h, f respectively; take an arbitrary point B, distance $OB = \gamma$; and taking m arbitrary, determine l, n by the equations

$$l^2 = m^2 + h^2 - \gamma^2,\ n^2 = m^2 + f^2 - \gamma^2;$$

then drawing through B an arbitrary line to meet the circles in A, C respectively; also describing a circle, centre B and radius $= m$; and through O drawing a line perpendicular to ABC to meet the last-mentioned circle in two points P, Q: for these points, the distances from the points A, B, C are $= l, m, n$ respectively.

To verify this, take O as the origin, OB for the axis of x, θ the inclination of ABC to this axis, $BA = r'$, $BC = r$; the coordinates of C, B, A are

$$\begin{array}{ll} \gamma + r\cos\theta, & r\sin\theta, \\ \gamma\quad, & 0\quad, \\ \gamma + r'\cos\theta, & -r'\sin\theta, \end{array}$$

whence, taking (x, y) for the coordinates of P (or Q), the equations to be verified are

$$\begin{aligned} (x - \gamma - r\cos\theta)^2 + (y - r\sin\theta)^2 &= n^2, \\ (x-\gamma)^2 + y^2 &= m^2, \\ (x - \gamma + r'\cos\theta)^2 + (y + r'\sin\theta)^2 &= l^2. \end{aligned}$$

By means of the second equation, the other two become

$$\begin{aligned} -2(x-\gamma) r\cos\theta - 2yr\sin\theta + r^2 &= n^2 - m^2, \\ 2(x-\gamma) r'\cos\theta + 2yr'\sin\theta + r'^2 &= l^2 - m^2; \end{aligned}$$

or, substituting for $n^2 - m^2$, $l^2 - m^2$ the values $f^2 - \gamma^2$ and $h^2 - \gamma^2$, the equations are

$$\begin{aligned} -2xr\cos\theta - 2yr\sin\theta + r^2 + 2\gamma r\cos\theta + \gamma^2 - f^2 &= 0, \\ 2xr'\cos\theta + 2yr'\sin\theta + r'^2 - 2\gamma r'\cos\theta + \gamma^2 - h^2 &= 0, \end{aligned}$$

viz. in virtue of the equations of the two circles, these reduce themselves each of them to

$$x\cos\theta + y\sin\theta = 0,$$

which equation, together with the second equation

$$(x-\gamma)^2 + y^2 = m^2,$$

determine (x, y) as above.

Reverting to the case where the locus of A is the circle

$$r'^2 - 2\gamma r' \cos\theta + \gamma^2 - h^2 = 0,$$

this gives

$$r' = \gamma\cos\theta + \sqrt{(h^2 - \gamma^2 \sin^2\theta)},$$

$$\frac{1}{r'} = \frac{\gamma\cos\theta - \sqrt{(h^2 - \gamma^2\sin^2\theta)}}{\gamma^2 - h^2};$$

so that for the transformed curve we have

$$r^2 + r\left(1 - \frac{l^2 - m^2}{\gamma^2 - h^2}\right)\gamma\cos\theta + r\left(1 + \frac{l^2 - m^2}{\gamma^2 - h^2}\right)\sqrt{(h^2 - \gamma^2\sin^2\theta)} + m^2 - n^2 = 0.$$

Putting for shortness $\dfrac{l^2 - m^2}{\gamma^2 - h^2} = \lambda$, and for r, $r\cos\theta$, $r\sin\theta$, writing $\sqrt{(x^2+y^2)}$, x, y respectively, this is

$$x^2 + y^2 + (1-\lambda)\gamma x + (1+\lambda)\sqrt{\{h^2(x^2+y^2) - \gamma^2 y^2\}} + m^2 - n^2 = 0,$$

or, what is the same thing,

$$\{x^2 + y^2 + (1-\lambda)\gamma x + m^2 - n^2\}^2 = (1+\lambda)^2\{h^2(x^2+y^2) - \gamma^2 y^2\},$$

a bicircular quartic. In the case $\lambda = -1$, it reduces itself to the circle

$$x^2 + y^2 + 2\gamma x + m^2 - n^2 = 0$$

twice, which is the case considered above; and in the case $\lambda = 1$, or $l^2 + h^2 = m^2 + \gamma^2$, the equation is

$$(x^2 + y^2 + m^2 - n^2)^2 = 4\{h^2(x^2+y^2) - \gamma^2 y^2\},$$

so that the curve is symmetrical in regard to each axis. In the case $\gamma = 0$, the locus is a pair of concentric circles, centre B.

The equation

$$\{x^2 + y^2 + (1-\lambda)\gamma x + m^2 - n^2\}^2 = (1+\lambda)^2\{h^2(x^2+y^2) - \gamma^2 y^2\},$$

which contains the four constants λ, γ, h and $m^2 - n^2$, may be written in the form

$$(x^2 + y^2 + Ax + B)^2 = ax^2 + ey^2,$$

(where the constants A, B, a, e are also arbitrary). This is, in fact, the equation of the general symmetrical bicircular quartic, referred to a properly-selected point on the axis as origin, viz. the origin is the centre of any one of the three involutions formed by the vertices (or points on the axis); say it is any one of the three involution-centres of the curve.

To show this, assume

$$(x-\alpha)(x-\beta)(x-\gamma)(x-\delta) = x^4 - px^3 + qx^2 - rx + s:$$

then, taking B arbitrary, the equation of the symmetrical bicircular quartic having for vertices the points $x=\alpha$, $x=\beta$, $x=\gamma$, $x=\delta$, is

$$(x^2+y^2-\tfrac{1}{2}px+B)^2=(2B+\tfrac{1}{4}p^2-q)\,x^2+(r-pB)\,x+(-s+B^2)\,;$$

in fact, this is the form of the general equation, and writing therein $y=0$, it becomes $x^4-px^3+qx^2-rx+s=0$, that is, $(x-\alpha)(x-\beta)(x-\gamma)(x-\delta)=0$. Hence, writing for convenience

$$\begin{aligned} A &= -\tfrac{1}{2}p, \\ a &= \ \ 2B+\tfrac{1}{4}p^2-q, \\ b &= \ \ r-pB, \\ c &= -s+B^2, \end{aligned}$$

the equation is

$$(x^2+y^2+Ax+B)^2=ax^2+bx+c.$$

This may be written

$$(x^2+y^2+Ax+B+\theta)^2=(a+2\theta)\,x^2+2\theta y^2+(b+2\theta A)\,x+c+2B\theta+\theta^2,$$

viz. assuming $\theta=-\dfrac{b}{2A}$ in order on the right-hand side to destroy the term in x, the equation is

$$\left(x^2+y^2+Ax+B-\frac{b}{2A}\right)^2=\left(a-\frac{b}{A}\right)x^2-\frac{b}{A}\,y^2+\frac{1}{4A^2}(b^2-4ABb+4A^2c),$$

which is of the form

$$(x^2+y^2+Ax+B)^2=ax^2+ey^2+f\,;$$

and if $f=0$, that is, if $b^2-4ABb+4A^2c=0$, then it is of the required form

$$(x^2+y^2+Ax+B)^2=ax^2+ey^2.$$

We have

$$\begin{aligned} b^2-4ABb+4A^2c &= (r-pB)^2+2pB\,(r-pB)+p^2\,(-s+B^2) \\ &= r^2-p^2s, \end{aligned}$$

or the required condition is $r^2-p^2s=0$. But we have

$$p^2s-r^2=(\alpha\delta-\beta\gamma)(\beta\delta-\gamma\alpha)(\gamma\delta-\alpha\beta),$$

as is easily verified by writing

$$p=\delta+p_0,\quad q=\delta p_0+q_0,\quad r=\delta q_0+r_0,\quad s=\delta r_0,$$

where p_0, q_0, r_0 stand for

$$\alpha+\beta+\gamma,\quad \beta\gamma+\gamma\alpha+\alpha\beta,\quad \alpha\beta\gamma,$$

respectively. The required condition thus is

$$(\alpha\delta-\beta\gamma)(\beta\delta-\gamma\alpha)(\gamma\delta-\alpha\beta)=0,$$

viz. the origin (that is, the fixed point B of the cell) must be at one of the three involution-centres.

Comparing the equation

$$\{x^2+y^2+(1-\lambda)\gamma x+m^2-n^2\}^2=(1+\lambda)^2\{h^2(x^2+y^2)-\gamma^2y^2\}$$

with the equation

$$\{x^2+y^2+Ax+B\}^2=ax^2+ey^2,$$

we have

$$A=(1-\lambda)\gamma,$$

$$B=m^2-n^2,$$

$$a=(1+\lambda)^2h^2,$$

$$e=(1+\lambda)^2(h^2-\gamma^2),$$

and thence $a-e=(1+\lambda)^2\gamma^2$. Consequently $\frac{a-e}{A^2}=\left(\frac{1+\lambda}{1-\lambda}\right)^2$, which gives λ: and then $h^2=\frac{a}{(1+\lambda)^2}$, $\gamma^2=\frac{a-e}{(1+\lambda)^2}$, $m^2-n^2=B$; viz. we thus have the values of λ, h, γ and m^2-n^2 for the description of a given curve $(x^2+y^2+Ax+B)^2=ax^2+ey^2$. In order that the description may be possible, a and $a-e$ must be each of them positive.

For the Cartesian a is $=e$, whence $1+\lambda=0$, and the equation becomes

$$(x^2+y^2+2\gamma x+m^2-n^2)^2=0,$$

which is a twice repeated circle; hence the Cartesian cannot be constructed by means of a cell as above.

To obtain a construction of the Cartesian, it may be remarked that, if a symmetrical bicircular quartic be inverted in regard to an axial focus, viz. if the focus be taken as the centre of inversion, we obtain a Cartesian. The axial foci of the curve

$$(x^2+y^2+Ax+B)^2=ax^2+ey^2$$

are points on the axis, the abscissa $x=\theta$ being determined by the equation

$$e(\theta^2+A\theta+B)^2-a(\theta^2-B)^2-ae\theta^2=0.$$

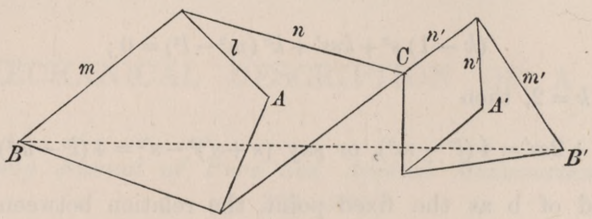
The equation referred to a focus as origin is therefore

$$\{x^2+y^2+(A+2\theta)x+B+\theta^2\}^2=ax^2+ey^2+2a\theta x+\theta^2;$$

then inverting, viz. for x, y writing $\frac{k^2x}{r^2}$, $\frac{k^2y}{r^2}$ (k arbitrary), we have, as may be verified the equation of a Cartesian.

The inversion can be performed mechanically by an ordinary Peaucellier-cell; the complete apparatus for the construction of a Cartesian is therefore as in fig. 2, viz. we have a cell BAC as before, B a fixed point, locus of A a circle (for convenience of drawing, the arrangement has been made BAC instead of ABC), and we connect with C a Peaucellier-cell $CA'B'$, arms n', n', m', the fixed point B' being on the axis, which is the line joining B with the centre of the circle described by A. This being so, then A describing a circle, C will describe a symmetrical bicircular quartic, and A' will describe the inverse of this, being in general a like curve; but if the position of B' be properly determined, viz. if B' be at a focus of the first-mentioned quartic,

Fig. 2.



then A' will describe a Cartesian. A further investigation would be necessary in order to determine how to adapt the apparatus to the description of a given Cartesian.

A more convenient mechanical description of a Cartesian is, however, that given in the paper which follows the present one [618].

The equation

$$\{x^2 + y^2 + (1 - \lambda)\gamma x + m^2 - n^2\}^2 = (1 + \lambda)^2 \{h^2(x^2 + y^2) - \gamma^2 y^2\}$$

may also be written

$$\{x^2 + y^2 + (1 - \lambda)\gamma x - \tfrac{1}{2}(1 + \lambda)^2 (h^2 - \gamma^2) + m^2 - n^2\}^2$$
$$= (1 + \lambda)^2 \{\gamma^2 x^2 - (1 - \lambda)(h^2 - \gamma^2)\gamma x + \tfrac{1}{4}(1 + \lambda)^2 (h^2 - \gamma^2)^2 - (m^2 - n^2)(h^2 - \gamma^2)\},$$

viz. the equation is now brought into the form

$$(x^2 + y^2 + Ax + B)^2 = ax^2 + bx + c.$$

Expressing the coefficients A, B, a, b, c in terms of λ, γ, h, $m^2 - n^2$, it appears by what precedes, that we should have identically $b^2 - 4ABb + 4A^2c = 0$, viz. this is the equation which expresses that the origin is an involution-centre.

If, instead of the original cell, we consider a new cell obtained by substituting for the arms PB, BQ, the arms pb, bq, jointed on to the points p, q on the arms CP, CQ respectively, and instead of B, making b the fixed point; then writing $C\mathrm{p} = kn$, $\mathrm{pb} = km$, so that the parameters of the cell are l, m, n, k, and taking $C\mathrm{b} = s$, $\mathrm{b}A = s'$, we have $s = kr$, $s + s' = r + r'$, that is, $r = \frac{s}{k}$, $r' = \frac{k-1}{k}s + s'$. Substituting in the equation between r, r', written for greater convenience in the form

$$(r + r')(rr' + m^2 - l^2) + (l^2 - n^2) r' = 0,$$

the relation between s, s' is found to be

$$(s + s')\left(\frac{k-1}{k^2}s^2 + \frac{ss'}{k} + m^2 - l^2\right) + (l^2 - n^2)\left(\frac{k-1}{k}s + s'\right) = 0.$$

On account of the term in s^3, this equation in its general form does not, it would appear, give rise to transformations of much elegance. If, however, $l = n$, then the relation becomes

$$(k-1)s^2 + kss' + k^2(m^2 - l^2) = 0\,;$$

and in particular, if $k = 2$, then

$$s^2 + 2ss' = 4(l^2 - m^2), \text{ or say } (s + s')^2 - s'^2 = 4(l^2 - m^2),$$

viz. taking A instead of b as the fixed point, the relation between the radii AC, Ab is $\rho^2 - \rho'^2 = 4(l^2 - m^2)$; the cell is in this case Sylvester's "quadratic-binomial extractor."