622.

ON A SYSTEM OF EQUATIONS CONNECTED WITH MALFATTI'S PROBLEM.

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I CONSIDER the equations

$$X, = by^2 + cz^2 - 2fyz - a(bc - f^2), = 0,$$
$$Y, = cz^2 + ax^2 - 2gzx - b(ca - g^2), = 0,$$
$$Z, = ax^2 + by^2 - 2hxy - c(ab - h^2), = 0,$$

where the constants (a, b, c, f, g, h) are such that

$$K, = abc - af^2 - bg^2 - ch^2 + 2fgh, = 0.$$

Hence, writing as usual (A, B, C, F, G, H) to denote the inverse coefficients

$$(bc - f^2,\ ca - g^2,\ ab - h^2,\ gh - af,\ hf - bg,\ fg - ch),$$

we have $(A, B, C, F, G, H \between x, y, z)^2 =$ the square of a linear function, $= (\alpha x + \beta y + \gamma z)^2$ suppose; that is,

$$(A, B, C, F, G, H) = (\alpha^2, \beta^2, \gamma^2, \beta\gamma, \gamma\alpha, \alpha\beta).$$

It is to be shown that the three quadric surfaces $X = 0$, $Y = 0$, $Z = 0$ intersect in a conic Θ lying in the plane $a\alpha x + b\beta y + c\gamma z = 0$, and in two points I, J; or more completely, that

the surfaces Y, Z meet in the conic Θ and a conic P,

" Z, X " " " Q,

" X, Y " " " R,

where the conics P, Q, R each pass through the two points I, J, and meet the conic Θ in two points, viz.,

the conics P, Θ meet in two points P_1, P_2,
" Q, Θ " " Q_1, Q_2,
" R, Θ " " R_1, R_2.

For this purpose, writing

$$\nabla = Aa - Ff, = Bb - Gg, = Cc - Hh, = abc - fgh,$$
$$\Omega = \tfrac{1}{2}(X + Y + Z), = ax^2 + by^2 + cz^2 - fyz - gzx - hxy - \nabla,$$
$$\theta = a\alpha x + b\beta y + c\gamma z,$$
$$\xi = \frac{Aa}{\alpha}x + \frac{Ff}{\beta}y + \frac{Ff}{\gamma}z,$$
$$\eta = \frac{Gg}{\alpha}x + \frac{Bb}{\beta}y + \frac{Gg}{\gamma}z,$$
$$\zeta = \frac{Hh}{\alpha}x + \frac{Hh}{\beta}y + \frac{Cc}{\gamma}z,$$

then we have identically

$$aA\Omega - \nabla X = \theta\xi,$$
$$bB\Omega - \nabla Y = \theta\eta,$$
$$cC\Omega - \nabla Z = \theta\zeta.$$

In fact, the first of these equations, written at full length, is

$$aA(ax^2 + by^2 + cz^2 - fyz - gzx - hxy - \nabla) - \nabla(by^2 + cz^2 - 2fyz - aA)$$
$$= (a\alpha x + b\beta y + c\gamma z)\left(\frac{Aa}{\alpha}x + \frac{Ff}{\beta}y + \frac{Ff}{\gamma}z\right),$$

where on the left-hand side the constant term is $= 0$. Comparing, first, the coefficients of x^2, y^2, z^2, on the two sides respectively, these are Aa^2, $(Aa - \nabla)b$, $(Aa - \nabla)c$, and Aa^2, Ffb, Ffc, which are equal. Comparing the coefficients of yz, zx, xy, the equations which remain to be verified are

$$-(aA - 2\nabla)f = Ff\left(c\frac{\gamma}{\beta} + b\frac{\beta}{\gamma}\right),$$
$$-aAg \qquad = Faf\frac{\alpha}{\gamma} + Aac\frac{\gamma}{\alpha},$$
$$-aAh \qquad = Faf\frac{\alpha}{\beta} + Aab\frac{\beta}{\alpha};$$

or, as these may be written,

$$-(aA - 2\nabla)\beta\gamma = F(c\gamma^2 + b\beta^2),$$
$$-\ Ag\gamma\alpha \qquad = Ff\alpha^2 + Ac\gamma^2,$$
$$-\ Ah\alpha\beta \qquad = Ff\alpha^2 + Ab\beta^2;$$

and, substituting for α^2, β^2, γ^2, $\beta\gamma$, $\gamma\alpha$, $\alpha\beta$ their values, these may be verified without difficulty.

It thus appears that the equations of the three quadric surfaces may be written in the form

$$aA\Omega - \theta\xi = 0,\ bB\Omega - \theta\eta = 0,\ cC\Omega - \theta\zeta = 0;$$

and we thus obtain the conics Θ, P, Q, R as the intersections of the surface $\Omega = 0$ by the four planes

$$\theta = 0,\quad \frac{\eta}{Bb} - \frac{\zeta}{Cc} = 0,\quad \frac{\zeta}{Cc} - \frac{\xi}{Aa} = 0,\quad \frac{\xi}{Aa} - \frac{\eta}{Bb} = 0,$$

respectively. There is no difficulty in verifying that the conics intersect as mentioned above, and that the coordinates of their points of intersection are

$$P,\ P_1 : \left(\sqrt{bc},\ \frac{ch}{\sqrt{bc}},\ \frac{bg}{\sqrt{bc}}\right),\ \left(-\sqrt{bc},\ -\frac{ch}{\sqrt{bc}},\ -\frac{bg}{\sqrt{bc}}\right);$$

$$Q,\ Q_1 : \left(\frac{ch}{\sqrt{ca}},\ \sqrt{ca},\ \frac{af}{\sqrt{ca}}\right),\ \left(-\frac{ch}{\sqrt{ca}},\ -\sqrt{ca},\ -\frac{af}{\sqrt{ca}}\right);$$

$$R,\ R_1 : \left(\frac{bg}{\sqrt{ab}},\ \frac{af}{\sqrt{ab}},\ \sqrt{ab}\right),\ \left(-\frac{bg}{\sqrt{ab}},\ -\frac{af}{\sqrt{ab}},\ -\sqrt{ab}\right);$$

$$I,\ J\ : (f,\ g,\ h),\qquad (-f,\ -g,\ -h).$$

In a paper "On a system of Equations connected with Malfatti's Equation and on another Algebraical System," *Camb. and Dublin Math. Journal*, vol. IV. (1849), pp. 270—275, [79], I considered a system of equations which, writing therein $\theta = 1$, and changing the signs of $(f,\ g,\ h)$, are the equations here considered, $X = 0$, $Y = 0$, $Z = 0$: only the constants $(a,\ b,\ c,\ f,\ g,\ h)$ are not connected by the equation $K = 0$, but are perfectly arbitrary. The three quadric surfaces intersect therefore in 8 points, the coordinates of which are obtained in the paper just referred to, viz. making the above changes of notation, the values are

$$x^2 = \frac{1}{2a}(abc + fgh - f\sqrt{BC} + g\sqrt{CA} + h\sqrt{AB}),$$

$$y^2 = \frac{1}{2b}(abc + fgh + f\sqrt{BC} - g\sqrt{CA} + h\sqrt{AB}),$$

$$z^2 = \frac{1}{2c}(abc + fgh + f\sqrt{BC} + g\sqrt{CA} - h\sqrt{AB}),$$

$$yz = \tfrac{1}{2}(gh + af + \sqrt{BC}),$$

$$zx = \tfrac{1}{2}(hf + bg + \sqrt{CA}),$$

$$xy = \tfrac{1}{2}(fg + ch + \sqrt{AB});$$

where the radicals are such that $\sqrt{BC}\,.\sqrt{CA}\,.\sqrt{AB} = ABC$, so that the system $(x^2,\ y^2,\ z^2,\ yz,\ zx,\ xy)$ has four values only, and consequently $(x,\ y,\ z)$ has eight values.

It is very remarkable that, introducing the foregoing relation $K=0$, there is not in the solution any indication that the intersection has become a conic and two points, but the solution gives eight determinate points, viz. the before-mentioned points P, P_1, Q, Q_1, R, R_1, and I, J.

To develope the solution, remark that, in virtue of the relation in question, we have

$$\sqrt{BC} = \pm F, \quad \sqrt{CA} = \pm G, \quad \sqrt{AB} = \pm H,$$

where the signs must be such that the product is $= FGH$ (viz. they must be all positive, or else one positive and the other two negative); for, taking the product to be $+FGH$, the equations give

$$0 = ABC - FGH,$$

that is,

$$0 = A(BC - F^2) - F(GH - AF), \ = K(Aa - Ff), \ = K\nabla,$$

which is true in virtue of the relation $K=0$. Taking the signs all positive, we have for $x^2, y^2, z^2, yz, zx, xy$, the values $f^2, g^2, h^2, gh, hf, fg$, viz. we have thus the points

$$(f, g, h), \quad (-f, -g, -h),$$

which are the points I, J. Taking the signs one positive and the other two negative, say $\sqrt{BC} = F$, $\sqrt{CA} = -G$, $\sqrt{AB} = -H$, we find for $x^2, y^2, z^2, yz, zx, xy$ the values $bc, \frac{ch^2}{b}, \frac{bg^2}{c}, gh, bg, ch$, viz. we have thus the points

$$\left(\sqrt{bc}, \frac{ch}{\sqrt{bc}}, \frac{bg}{\sqrt{bc}}\right), \quad \left(-\sqrt{bc}, -\frac{ch}{\sqrt{bc}}, -\frac{bg}{\sqrt{bc}}\right),$$

which are the points P, P_1; and the other two combinations of sign give of course the points Q, Q_1 and R, R_1 respectively.

If the coefficients (a, b, c, f, g, h), instead of the foregoing relation $K=0$, satisfy the relation

$$abc - af^2 - bg^2 - ch^2 - 2fgh = 0, \text{ say } K' = 0,$$

the quadric surfaces intersect in 8 points, the coordinates of which are given by the general formulæ: but the expressions assume a very simple form. Writing for shortness

$$F' = gh + af, \quad G' = hf + bg, \quad H' = fg + ch,$$

then, in virtue of the assumed relation,

$$\sqrt{BC} = \pm F', \quad \sqrt{CA} = \pm G', \quad \sqrt{AB} = \pm H',$$

where the signs are such that the product of the three terms is positive, viz. they must be all positive, or else one positive and the other two negative. For, assuming it to be so, we have

$$0 = ABC - F'G'H',$$

that is,

$$0 = A(BC - F'^2) - F'(G'H' - AF'),$$
$$= K'(Aa + F'f), \; = K'(abc + fgh);$$

which is right, in virtue of the relation $K' = 0$. Taking the signs all positive, we find for $(x^2, y^2, z^2, yz, zx, xy)$ the values (A, B, C, F', G', H'), giving two points of intersection

$$\left(\sqrt{A},\ \frac{H'}{\sqrt{A}},\ \frac{G'}{\sqrt{A}}\right) \text{ and } \left(-\sqrt{A},\ -\frac{H'}{\sqrt{A}},\ -\frac{G'}{\sqrt{A}}\right).$$

Taking the signs one positive and the other two negative, say

$$\sqrt{BC} = F', \quad \sqrt{CA} = -G', \quad \sqrt{AB} = -H',$$

we find for $(x^2, y^2, z^2, yz, zx, xy)$ the values

$$\left(0,\ \frac{Cc}{b},\ \frac{Bb}{c},\ F',\ 0,\ 0\right),$$

viz. we have thus two intersections

$$\left(0,\ \sqrt{\frac{Cc}{b}},\ F'\sqrt{\frac{b}{Cc}}\right),\ \left(0,\ -G'\sqrt{\frac{Cc}{b}},\ -F'\sqrt{\frac{b}{Cc}}\right);$$

and the other combinations of signs give the remaining two pairs of intersections

$$\left(G'\sqrt{\frac{c}{Aa}},\ 0,\ \sqrt{\frac{Aa}{c}}\right),\ \left(-G'\sqrt{\frac{c}{Aa}},\ 0,\ -\sqrt{\frac{Aa}{c}}\right),$$

and

$$\left(\sqrt{\frac{Bb}{a}},\ H'\sqrt{\frac{a}{Bb}},\ 0\right),\ \left(-\sqrt{\frac{Bb}{a}},\ -H'\sqrt{\frac{a}{Bb}},\ 0\right).$$

But the most convenient statement of the result is that the values of $(ax^2, by^2, cz^2, yz, zx, xy)$, for the four pairs of points respectively, are

$$\begin{array}{l}(aA,\ bB,\ cC,\ F',\ G',\ H'),\\ (\ 0\ ,\ cC,\ bB,\ F',\ 0,\ 0\ ),\\ (cC,\ 0\ ,\ aA,\ 0,\ G',\ 0\ ),\\ (bB,\ aA,\ 0\ ,\ 0,\ 0,\ H');\end{array}$$

there is no difficulty in substituting these values in the original equations, and in verifying that the equations are in each case satisfied.