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ON THE BICURSAL SEXTIC.

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IN the paper "On the mechanical description of certain sextic curves," *Proceedings of the London Mathematical Society*, vol. IV. (1872), pp. 105—111, [504], I obtained the bicursal sextic as a rational transformation of a binodal quartic. The theory was in effect as follows: taking Ω, P, Q, R, each of them a function of λ, μ of the form $(*\between 1, \lambda)^2 (1, \mu)^2$, and considering (λ, μ) as connected by the equation $\Omega = 0$, (viz. λ, μ being coordinates, this represents a binodal quartic), then, if we assume $x : y : z = P : Q : R$, the locus of the point (x, y, z) is a curve rationally connected with the binodal quartic, viz. the points of the two curves have with each other a $(1, 1)$ correspondence; whence the locus in question, say the curve $U = 0$, is bicursal. The degree is obtained as the number of the intersections of the curve by an arbitrary line, or, what is the same thing, the number of the variable intersections of the corresponding $\lambda\mu$-curves

$$\Omega = 0, \quad \alpha P + \beta Q + \gamma R = 0,$$

viz. each of these being a quartic curve having the same two nodes, the nodes each count as 4 intersections, and the number of the remaining intersections is $4 . 4 - 2 . 4, = 8$, and thus the curve $U = 0$ is in general of the order 8. But if the curves $\Omega = 0$, $P = 0$, $Q = 0$, $R = 0$ have (besides the nodes) k common intersections, then these are also fixed intersections of the two curves $\Omega = 0$, $\alpha P + \beta Q + \gamma R = 0$, and the number of variable intersections is reduced to $8 - k$; we have thus $8 - k$ as the order of the curve $U = 0$. In particular, if $k = 2$, then the curve is a bicursal sextic.

The theory assumes a different and more simple form if, in the several functions Ω, P, Q, R, we suppose that the terms in λ^2, μ^2 are wanting. The curves $\Omega = 0$, $P = 0$, $Q = 0$, $R = 0$ are here cubics having two common points; the curve $U = 0$, *quà* rational transformation of the cubic $\Omega=0$, is still a bicursal curve; but its order is given as the number of the variable intersections of the cubics

$$\Omega=0, \quad \alpha P+\beta Q+\gamma R=0,$$

viz. this is $=3.3-2, =7$. But if the curves $\Omega=0$, $P=0$, $Q=0$, $R=0$ have (besides the before-mentioned two common points) k other common points, then the number of the variable intersections is $=7-k$: and this is therefore the order of the curve $U=0$. In particular, if $k=1$, then the curve is a bicursal sextic. And, in the present paper, I consider the binodal sextic as thus obtained, viz. as given by the equations $\Omega=0$, $x:y:z=P:Q:R$, where $\Omega=0$, $P=0$, $Q=0$, $R=0$ are cubics, having (in all) three common points.

The bicursal sextic has in general 9 nodes; but 3 of these may unite together into a triple point: this will be the case if, in the series of curves $\alpha P+\beta Q+\gamma R=0$, there are any two curves which have 3 common intersections with the curve $\Omega=0$. (Observe that we throughout disregard the 3 common points of the curves $\Omega=0$, $P=0$, $Q=0$, $R=0$, and attend only to the 6 variable points of intersection of the curves $\Omega=0$ and $\alpha P+\beta Q+\gamma R=0$,—the meaning is, that there are two curves of the series such that, attending only to the 6 variable intersections of each of them with the curve $\Omega=0$, there are three common intersections.) For, supposing the two curves to be $\alpha P+\beta Q+\gamma R=0$ and $\alpha' P+\beta' Q+\gamma' R=0$, then any curve whatever

$$\alpha P+\beta Q+\gamma R+\theta(\alpha' P+\beta' Q+\gamma' R)=0$$

has the same three intersections with the curve $\Omega=0$, say these are the points A_1, A_2, A_3, the coordinates of which are independent of θ. Hence the line

$$(\alpha x+\beta y+\gamma z)+\theta(\alpha' x+\beta' y+\gamma' z)=0$$

intersects the curve $U=0$ in six points, three of which, as corresponding to the points A_1, A_2, A_3, are independent of θ, viz. they are the same three points for any line whatever of the series; and this means that the curve $U=0$ has at the point

$$(\alpha x+\beta y+\gamma z=0, \quad \alpha' x+\beta' y+\gamma' z=0)$$

a triple point; and that to this triple point correspond the three points A_1, A_2, A_3.

We may, in the series of lines $\alpha x+\beta y+\gamma z+\theta(\alpha' x+\beta' y+\gamma' z)=0$, rationally determine θ so that one of the three variable points of intersection shall correspond to A_1, A_2, or A_3; viz. θ must be such that the curve $\alpha P+\beta Q+\gamma R+\theta(\alpha' P+\beta' Q+\gamma' R)=0$ shall touch the curve $\Omega=0$ at one of the points A_1, A_2, A_3. The three lines thus determined are the three tangents to the curve at the triple point: and the three branches may be considered as corresponding to the three points A_1, A_2, A_3, respectively.

There is no loss of generality in assuming that the triple point is the point $(x=0,\ z=0)$; the condition then simply is that the curves $P=0$, $R=0$ shall have three common intersections with the curve $\Omega=0$; and the tangents at the triple point are $x+\theta z=0$, θ being so determined that one of the three variable points of intersection shall correspond to one of the three points A_1, A_2, A_3: in particular, if this is the case for the line $x=0$, then this line will be one of the tangents at the triple point.

The bicursal sextic may have a second triple point, viz. three other nodes may unite together into a triple point. The theory is precisely the same: we must have two other curves $\alpha P + \beta Q + \gamma R = 0$, $\alpha' P + \beta' Q + \gamma' R = 0$, having with the curve $\Omega = 0$ three common intersections B_1, B_2, B_3: there is then a second triple point

$$(\alpha x + \beta y + \gamma z = 0, \quad \alpha' x + \beta' y + \gamma' z = 0);$$

and, to find the tangents at this point, we must determine θ so that one of the variable points of intersection of the line

$$\alpha x + \beta y + \gamma z + \theta (\alpha' x + \beta' y + \gamma' z) = 0$$

with the sextic shall correspond with B_1, B_2, or B_3; viz. θ must be such that the curve $\alpha P + \beta Q + \gamma R + \theta (\alpha' P + \beta' Q + \gamma' R) = 0$ shall touch the curve $\Omega = 0$ at one of the points B_1, B_2, B_3. In particular, if, as before, the curves $P = 0$, $R = 0$ have three common intersections with the curve $\Omega = 0$, and if, moreover, the curves $Q = 0$, $R = 0$ have three common intersections with the curve $\Omega = 0$, then the bicursal sextic will have the two triple points $(x = 0,\ z = 0)$ and $(y = 0,\ z = 0)$; and it may further happen that the line $x = 0$ is a tangent at the first triple point, and the line $y = 0$ a tangent at the second triple point. The sextic may in like manner have a third triple point, but this is a special case which I do not at present consider.

I write for greater convenience $\dfrac{\lambda}{\nu}$, $\dfrac{\mu}{\nu}$ in place of λ, μ, so as to make Ω, P, Q, R each of them a homogeneous cubic function of (λ, μ, ν); and I give to these functions, not the most general values belonging to a bicursal sextic with two triple points, but the values in the form obtained for them, as appearing further on, in the problem of three-bar motion; viz. the equations $\Omega = 0$, $x : y : z = P : Q : R$ are respectively taken to be

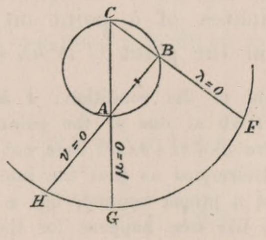
$$\nu (h\nu\lambda + f\lambda^2) + \mu (g\nu^2 + e\nu\lambda + g\lambda^2) + \mu^2 (f\nu + h\lambda) = 0,$$

$$x : y : z = \lambda\mu (a\lambda + b\mu) : \nu^2 (c\lambda + d\mu) : \lambda\mu\nu.$$

The four curves $\Omega = 0$, $P = 0$, $Q = 0$, $R = 0$ have thus the three common intersections

$$(\mu = 0,\ \nu = 0),\ (\nu = 0,\ \lambda = 0),\ (\lambda = 0,\ \mu = 0),$$

represented in the figure by the points A, B, C; the curve drawn in the figure is the curve $\Omega = 0$, and the points F, G, H are the third points of intersection of the cubic with the lines BC, CA, AB respectively.



The equation $P + \theta R = 0$ is here $\lambda\mu (a\lambda + b\mu + \theta\nu) = 0$, which intersects $\Omega = 0$ in the points C, A, G, C, B, F, and the three intersections by the line $a\lambda + b\mu + \theta\nu = 0$;

viz. excluding the fixed points A, B, C, the six intersections are C, F, G, and the three intersections by the line. Hence, of the six intersections, we have C, F, G independent of θ, or we have $(x=0,\ z=0)$ a triple point, say I, corresponding to the three points C, F, G, viz. these are the points

$$(\lambda,\ \mu,\ \nu)=(0,\ 0,\ 1),\quad (0,\ g,\ -f),\quad (h,\ 0,\ -f),\qquad (C,\ F,\ G).$$

The equation $Q+\theta R=0$ is $\nu(c\lambda\nu+d\mu\nu+\theta\lambda\mu)=0$; viz. the line $\nu=0$ meets $\Omega=0$ in the three points A, B, H, and the conic $c\lambda\nu+d\mu\nu+\theta\lambda\mu=0$ meets $\Omega=0$ in the points A, B, C, and three other points: hence, rejecting the points A, B, C, the six points of intersection are the points A, B, H, and the three variable points of intersection by the conic; or we have $(y=0,\ z=0)$ a triple point, say J, corresponding to the three points A, B, H, viz. these are the points

$$(\lambda,\ \mu,\ \nu)=(1,\ 0,\ 0),\quad (0,\ 1,\ 0),\quad (h,\ -g,\ 0),\qquad (A, B, H).$$

To find the tangents at the triple point I, these are $x+\theta z=0$, where θ is to be successively determined by the conditions that the line $a\lambda+b\mu+\theta\nu=0$ shall pass through the points C, F, G*; viz. we thus have

$$\theta=0,\qquad x=0,\ \text{the tangent corresponding to the point } C,\ (0,\ 0,\ 1),$$

$$\theta=+\frac{bg}{f},\quad fx+bgz=0,\quad \text{"}\quad \text{"}\quad \text{"}\quad \text{"}\quad F,\ (0,\ g,\ -f),$$

$$\theta=+\frac{ah}{f},\quad fx+ahz=0,\quad \text{"}\quad \text{"}\quad \text{"}\quad \text{"}\quad G,\ (h,\ 0,\ -f).$$

And similarly, at the triple point J, the tangents are $y+\theta z=0$, where θ is to be successively determined by the conditions that the conic $c\nu\lambda+d\nu\mu+\theta\lambda\mu=0$ shall pass through the point H, and shall touch the cubic at the points A, B; viz. we thus have

$$\theta=0,\qquad y=0,\ \text{the tangent corresponding to the point } H,\ (h,\ -g,\ 0),$$

$$\theta=\frac{cg}{f},\quad fy+cgz=0,\quad \text{"}\quad \text{"}\quad \text{"}\quad \text{"}\quad A,\ (1,\ 0,\ 0),$$

$$\theta=\frac{dh}{f},\quad fy+dhz=0,\quad \text{"}\quad \text{"}\quad \text{"}\quad \text{"}\quad B,\ (0,\ 1,\ 0).$$

The two last values of θ are obtained by the consideration that the equations of the tangents to $\Omega=0$ at the points A, B respectively, are $g\mu+f\nu=0$, $h\lambda+f\nu=0$, where λ, μ, ν are current coordinates of a point on the tangent: it may be added that the equation of the tangent at the point C is $h\lambda+g\mu=0$.

* Observe the somewhat altered form of the condition: θ is to be determined so that the cubic $\lambda\mu(a\lambda+b\mu+\theta\nu)=0$ shall touch the cubic $\Omega=0$ at one of the points C, F, G: but, as the first-mentioned cubic breaks up, and the component curve $a\lambda+b\mu+\theta\nu=0$ does not pass through any one of these points, this can only mean that θ shall be so determined as that the line shall pass through one of these points, viz. that there shall be at the point, not a proper contact, but a double intersection, arising from a node of the cubic $\lambda\mu(a\lambda+b\mu+\theta\nu)=0$. And the like case happens for the other triple point; viz. there the cubic $\nu(c\nu\lambda+d\nu\mu+\theta\lambda\mu)=0$ is to touch the cubic $\Omega=0$ at one of the points A, B, H; the component conic $c\nu\lambda+d\nu\mu+\theta\lambda\mu=0$ passes through the points A and B but not through H; hence the conditions for θ are, that the conic shall touch the cubic at A or B, or that it shall pass through H.

The three-bar curve may be represented by means of a system of equations of the last-mentioned form, viz. $x : y : z = \lambda\mu(a\lambda + b\mu) : \nu^2(c\lambda + d\mu) : \lambda\mu\nu$, where λ, μ, ν are connected as above; or, taking X, Y as ordinary rectangular coordinates, x, y, and z are here the circular coordinates $\frac{x}{z} = X + iY$, $\frac{y}{z} = X - iY$, and $z = 1$; and the parameters $\frac{\lambda}{\nu}$, $\frac{\mu}{\nu}$ denote like functions $\cos\theta + i\sin\theta$, $\cos\phi + i\sin\phi$ of angles which are the inclinations of two bars to a fixed line. Using, for convenience, Figure 2 of my paper on Three-bar Motion, (p. 553 of this volume), the curve is considered as the locus of the vertex O of the triangle OC_1B_1, connected by the bars C_1C and B_1B with the fixed points B and C respectively; and we have $CC_1 = a_2$, $OC_1 = b_1$, $OB_1 = c_1$, $C_1B_1 = a_1$, $B_1B = a_3$. Also, to avoid confusion with the foregoing notation of the present paper, instead of calling it a, I take $BC = a_0$: the angle OC_1B_1 is $= C_1$, and $\cos C_1 + i\sin C_1$ is taken $= \gamma$.

Hence, taking the origin at C, the axis of X coinciding with CB and that of Y being at right angles to it: taking also θ, ϕ, ψ for the inclinations of CC_1, C_1B_1, and B_1B to CB, we have

$$\begin{aligned} a_2\cos\theta + a_1\cos\phi - a &= -a_3\cos\psi, \\ a_2\sin\theta + a_1\sin\phi \quad &= \quad a_3\sin\psi; \end{aligned}$$

viz. writing $\cos\theta + i\sin\theta = \lambda$, $\cos\phi + i\sin\phi = \mu$, these give

$$\begin{aligned} a_2\lambda + a_1\mu - a_0 &= -a_3(\cos\psi - i\sin\psi), \\ a_2\frac{1}{\lambda} + a_1\frac{1}{\mu} - a_0 &= -a_3(\cos\psi + i\sin\psi), \end{aligned}$$

that is,

$$(a_2\lambda + a_1\mu - a)\left(a_2\frac{1}{\lambda} + a_1\frac{1}{\mu} - a_0\right) - a_3^2 = 0;$$

viz.

$$(a_0^2 + a_1^2 + a_2^2 - a_3^2) + a_1a_2\left(\frac{\mu}{\lambda} + \frac{\lambda}{\mu}\right) - a_0a_2\left(\lambda + \frac{1}{\lambda}\right) - a_0a_1\left(\mu + \frac{1}{\mu}\right) = 0,$$

for the relation between the parameters λ, μ. And then

$$\begin{aligned} X &= a_2\cos\theta + b_1\cos(\phi + C_1), \\ Y &= a_2\sin\theta + b_1\sin(\phi + C_1); \end{aligned}$$

viz. if x, $y = X + iY$, $X - iY$, then

$$\begin{cases} x = a_2\lambda + b_2\gamma_1\mu, \\ y = a_2\dfrac{1}{\lambda} + b_2\dfrac{1}{\gamma_1\mu}, \end{cases}$$

which equations determine the coordinates (x, y) in terms of the parameters λ, μ connected by the foregoing relation.

Writing for homogeneity $\frac{\lambda}{\nu}$, $\frac{\mu}{\nu}$ in place of λ, μ, and $\frac{x}{z}$, $\frac{y}{z}$ in place of x, y, the equations become

$$(a_0^2+a_1^2+a_2^2-a_3^2)\lambda\mu\nu + a_1a_2(\lambda^2+\mu^2)\nu - a_0a_2\mu(\nu^2+\lambda^2) - a_0a_1\lambda(\mu^2+\nu^2) = 0,$$

and

$$x : y : z = (a_2\lambda + b_2\gamma_1\mu)\lambda\mu : \left(\frac{b_2}{\gamma_1}\lambda + a_2\mu\right)\nu^2 : \lambda\mu\nu.$$

Comparing with the foregoing equations

$$e\lambda\mu\nu + f(\lambda^2+\mu^2)\nu + g(\nu^2+\lambda^2)\mu + h(\mu^2+\nu^2)\lambda = 0,$$

and

$$x : y : z = (a\lambda + b\mu)\lambda\mu : (c\lambda + d\mu)\nu^2 : \lambda\mu\nu,$$

the equations agree together, and we have

$$\begin{cases} e = a_0^2+a_1^2+a_2^2-a_3^2, \\ f = +a_1a_2, \\ g = -a_0a_2, \\ h = -a_0a_1, \\ a = \quad a_2, \\ b = \quad b_1\gamma_1, \\ c = \quad \dfrac{b_1}{\gamma_1}, \\ d = \quad a_2. \end{cases}$$

The tangents at the triple points thus are

$$x = 0, \qquad y = 0,$$

$$a_1x - a_0b_1\gamma_1z = 0, \qquad a_1y - \frac{a_0b_1}{\gamma_1}z = 0,$$

$$x - a_0z = 0, \qquad y - a_0z = 0;$$

viz. restoring the rectangular coordinates, and for γ substituting the value $\cos C + i\sin C$, for a_0 writing a, and taking $b = \frac{ab_1}{a_1}$, we have

$$X + iY = 0, \qquad X - iY = 0,$$

$$X + iY = b(\cos C + i\sin C), \qquad X - iY = b(\cos C - i\sin C),$$

$$X + iY = a_0, \qquad X - iY = a_0;$$

viz. the first two intersect in the point $(0, 0)$, the second two in the point $(b\cos C, b\sin C)$, the third two in the point $(a, 0)$: the first and third of these are the points B and C, the second of them is the point A of the figure; viz. the formulæ give the point A, forming, with B and C, a triad of foci.