627.

GEOMETRICAL ILLUSTRATION OF A THEOREM RELATING TO AN IRRATIONAL FUNCTION OF AN IMAGINARY VARIABLE.

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If we have v, a function of u, determined by an equation $f(u, v) = 0$, then to any given imaginary value $x+iy$ of u there belong two or more values, in general imaginary, $x'+iy'$ of v: and for the complete understanding of the relation between the two imaginary variables, we require to know the series of values $x'+iy'$ which correspond to a given series of values $x+iy$, of v, u respectively. We must for this purpose take x, y as the coordinates of a point P in a plane Π, and x', y' as the coordinates of a corresponding point P' in another plane Π'. The series of values $x+iy$ of u is then represented by means of a curve in the first plane, and the series of values $x'+iy'$ of v by means of a corresponding curve in the second plane. The correspondence between the two points P and P' is of course established by the two equations into which the given equation $f(x+iy,\ x'+iy') = 0$ breaks up, on the assumption that x, y, x', y' are all of them real. If we assume that the coefficients in the equation are real, then the two equations are

$$f(x+iy,\ x'+iy') + f(x-iy,\ x'-iy') = 0,$$

$$f(x+iy,\ x'+iy') - f(x-iy,\ x'-iy') = 0;$$

viz. if in these equations we regard either set of coordinates, say (x, y), as constants, then the other set (x', y') are the coordinates of any real point of intersection of the curves represented by these equations respectively.

I consider the particular case where the equation between u, v is $u^2+v^2=a^2$: we have here $(x+iy)^2+(x'+iy')^2=a^2$: so that, to a given point P in the first plane, there correspond in general two points P_1', P_2' in the second plane: but to each of the points A and B, coordinates $(a, 0)$ and $(-a, 0)$, there corresponds only a single point in the second plane.

We have here a particular case of a well-known theorem: viz. if from a given point P we pass by a closed curve, not containing within it either of the points A or B, back to the initial point P, we pass in the other plane from P_1' by a closed curve back to P_1'; and similarly from P_2' by a closed curve back to P_2': but if the closed curve described by P contain within it A or B, then, in the other plane, we pass continuously from P_1' to P_2'; and also continuously from P_2' to P_1'.

The relations between (x, y), (x', y') are

$$x'^2 - y'^2 = a^2 - (x^2 - y^2),$$

$$x'y' = -xy,$$

whence also

$$(x'^2 + y'^2)^2 = a^4 - 2a^2(x^2 - y^2) + (x^2 + y^2)^2.$$

And if the point (x, y) describe a curve $x^2 + y^2 = \phi(x^2 - y^2)$, then will the point (x', y') describe a curve $x'^2 + y'^2 = \psi(x'^2 - y'^2)$, obtained by the elimination of $x^2 - y^2$ from the two equations

$$x'^2 - y'^2 = \quad a^2 - \quad (x^2 - y^2),$$

$$(x'^2 + y'^2)^2 = \quad a^4 - 2a^2(x^2 - y^2) + \phi(x^2 - y^2);$$

viz. this is

$$(x'^2 + y'^2)^2 = -a^4 + 2a^2(x'^2 - y'^2) + \phi\{a^2 - (x'^2 - y'^2)\}.$$

In particular, if the one curve be $(x^2 + y^2)^2 = \alpha + \beta(x^2 - y^2)$; then the other curve is

$$(x'^2 + y'^2)^2 = -a^4 + 2a^2(x'^2 - y'^2) + \alpha + \beta\{\alpha^2 - (x'^2 - y'^2)\},$$

that is,

$$(x'^2 + y'^2)^2 = \alpha' + \beta'(x'^2 - y'^2),$$

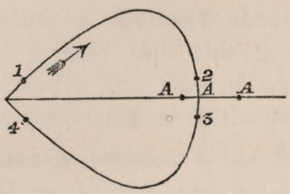
where

$$\alpha' = -a^4 + \beta a^2 + \alpha, \quad \beta' = 2a^2 - \beta.$$

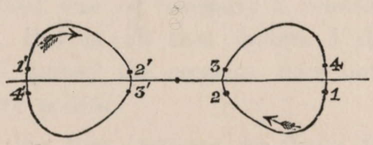
Writing for greater simplicity $a = 1$, then $\alpha' = -1 + \alpha + \beta$, $\beta' = 2 - \beta$; in particular, if $\alpha = 0$, then $\alpha' = -1 + \beta$, $\beta' = 2 - \beta$.

Supposing successively $\beta < 1$, $\beta = 1$, and $\beta > 1$, then in each case P describes a closed curve or half figure-of-eight, as shown in the annexed P-figure; but in the first case the point A is inside the curve, in the second case on it, and in the third case outside it, as shown by the letters A, A, A of the figure; and, corresponding to the three cases respectively, we have the three P'-figures, the curve in the first of them consisting of two ovals, in the second of them being a figure of eight, and in the third a twice-indented or pinched oval: the small figures 1, 2, 3, 4 in the P-figure, and 1, 2, 3, 4 and 1′, 2′, 3′, 4′ in the P'-figures serve to show the corresponding positions of the points P and P_1', P_2' respectively; and the courses are further indicated by the arrows. And we thus see how the two separate closed curves described

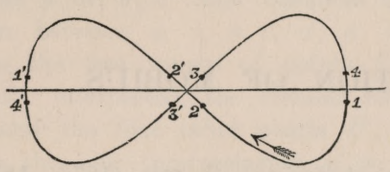
P-Figure.



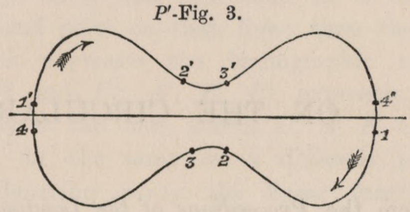
P'-Fig. 1.



P'-Fig. 2.



P'-Fig. 3.



by P_1' and P_2', as in figure 1, change into the single closed curve described one half of it by P_1' and the other half of it by P_2' as in figure 3.