629.

ON THE LINEAR TRANSFORMATION OF THE INTEGRAL $\int \frac{du}{\sqrt{U}}$.

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THE quartic function U is taken to be $= \epsilon . u - a . u - b . u - c . u - d$, where a, b, c, d are imaginary values represented in the usual manner by means of the points A, B, C, D; viz. if $a = \alpha_0 + \alpha_1 i$, then A is the point whose rectangular coordinates are α_0, α_1; and the like as regards B, C, D. And I consider chiefly the definite integrals such as $\int_a^b \frac{du}{\sqrt{U}}$ where the path is taken to be the right line from A to B. There is here nothing to fix the sign of the radical; but if at any particular point of the path we assign to it at pleasure one of its two values, then (the radical varying continuously) this determines the value at every other point of the path; and the integral defined as above is completely determinate except as to its sign, which might be fixed as above, but which is better left indeterminate. The integral, thus determinate except as to its sign, is denoted by (AB).

I wish to establish the theorem that, if the points A, B, C, D taken in this order form a convex quadrilateral, then

$$(AB) = \pm (CD), \quad (AD) = \pm (BC), \text{ but not } (AC) = \pm (BD);$$

whereas, if the four points form a triangle and interior point, then the three equations all hold good. I regard the theorem as the precise statement of Bouquet and Briot's theorem, $A - B + C - D = 0$, or say $(OA) - (OB) + (OC) - (OD) = 0$, where the four terms are the rectilinear integrals taken from a point O to the four points A, B, C, D respectively. The two cases may be called, for shortness, the convex and the reentrant cases respectively.

To prove in the case of a convex quadrilateral that (AC) is not $= \pm (BD)$, it is sufficient to consider the integral $\int \frac{du}{\sqrt{u^4-1}}$, where A, B, C, D are the points (1, 0), (0, 1), (−1, 0), and (0, −1) respectively, and where, writing $v = iu$, it at once appears that we have

$$\int_{-1}^{1} \frac{du}{\sqrt{u^4-1}} = \pm i \int_{-i}^{i} \frac{du}{\sqrt{u^4-1}},$$

that is,

$$(AC) = \pm i\,(BD), \text{ not } (AC) = \pm\,(BD).$$

But I consider the general question of the linear transformation. If a', b', c', d' correspond homographically to a, b, c, d, then to represent these values a', b', c', d' we have the points A', B', C', D', connected with A, B, C, D according to the circular relation of Möbius; and then, making u', a', b', c', d' correspond homographically to u, a, b, c, d, and representing in like manner the variables u, u' by the points U, U' respectively, we have the circular relation between the two systems U, A, B, C, D and U', A', B', C', D'.

Before going further I remark that the distinction of the convex and reentrant cases is not an invariable one; the figures are transformable the one into the other. Thus, taking C on the line BD (that is, between B and D, not on the line produced), there is not this relation between B', C', D', and the figure $A'B'C'D'$ is convex or reentrant as the case may be. Giving to C an infinitesimal displacement to the one side or the other of the line BD, we have in the one case a convex figure, in the other case a reentrant figure $ABCD$; but the corresponding displacement of C' being infinitesimal, the figure $A'B'C'D'$ remains for either displacement, convex or reentrant, as it originally was; that is, we have a convex figure $ABCD$ and a reentrant figure $ABCD$, each corresponding to the figure $A'B'C'D'$ (which is convex, or else reentrant, as the case may be).

Writing for convenience

$$\mathrm{a}, \mathrm{b}, \mathrm{c}, \mathrm{f}, \mathrm{g}, \mathrm{h} = b-c,\ c-a,\ a-b,\ a-d,\ b-d,\ c-d,$$

$$\mathrm{a}', \mathrm{b}', \mathrm{c}', \mathrm{f}', \mathrm{g}', \mathrm{h}' = b'-c',\ c'-a',\ a'-b',\ a'-d',\ b'-d',\ c'-d',$$

so that identically

$$\mathrm{af} + \mathrm{bg} + \mathrm{ch} = 0, \quad \mathrm{a'f'} + \mathrm{b'g'} + \mathrm{c'h'} = 0,$$

then the homographic relation between (a, b, c, d), (a', b', c', d') may be written in the forms

$$\mathrm{af} : \mathrm{bg} : \mathrm{ch} = \mathrm{a'f'} : \mathrm{b'g'} : \mathrm{c'h'},$$

or, what is the same thing, there exists a quantity N such that

$$\frac{\mathrm{a'f'}}{\mathrm{af}} = \frac{\mathrm{b'g'}}{\mathrm{bg}} = \frac{\mathrm{c'h'}}{\mathrm{ch}} = N^2.$$

The relation between u, u' may be written in the forms

$$\frac{u'-a'}{u'-d'} = P\frac{u-a}{u-d},\quad \frac{u'-b'}{u'-d'} = Q\frac{u-b}{u-d},\quad \frac{u'-c'}{u'-d'} = R\frac{u-c}{u-d};$$

and then, writing for u, u' their corresponding values, we find

$$P = \frac{\mathrm{b'h}}{\mathrm{bh'}} = \frac{\mathrm{c'g}}{\mathrm{cg'}},\quad Q = \frac{\mathrm{c'f}}{\mathrm{cf'}} = \frac{\mathrm{a'h}}{\mathrm{ah'}},\quad R = \frac{\mathrm{a'g}}{\mathrm{ag'}} = \frac{\mathrm{b'f}}{\mathrm{bf'}},$$

giving

$$\mathrm{f}^2PN^2 = \mathrm{f}'^2QR,\quad \mathrm{g}^2QN^2 = \mathrm{g}'^2RP,\quad \mathrm{h}^2RN^2 = \mathrm{h}'^2PQ,\quad \sqrt{PQR} = \frac{\mathrm{fgh}}{\mathrm{f'g'h'}}N^3.$$

Differentiating any one of the equations in (u, u'), for instance the first of them, we find

$$\frac{\mathrm{f}'du'}{(u'-d')^2} = \frac{\mathrm{f}Pdu}{(u-d)^2};$$

then, forming the equation

$$\frac{\sqrt{\epsilon\,.\,u'-a'\,.\,u'-b'\,.\,u'-c'\,.\,u'-d'}}{(u'-d')^2} = \pm\frac{\sqrt{PQR}\sqrt{\epsilon\,.\,u-a\,.\,u-b\,.\,u-c\,.\,u-d}}{(u-d)^2},$$

and attending to the relation $\mathrm{f}^2PN^2 = \mathrm{f}'^2QR$, we obtain

$$\pm\frac{Ndu'}{\sqrt{U'}} = \frac{du}{\sqrt{U}},$$

which is the differential relation between u, u'.

We have in connection with A, B, C, D the point O, and in connection with A', B', C', D' the point O'. As U describes the right line AB, U' describes the arc not containing O' of the circle $A'B'O'$; for observe that O' corresponds in the second figure to the point at infinity on the line AB, viz. as U passes from A to B, not passing through the point at infinity, U' must pass from A' to B', not passing through the point O', that is, it must describe, not the arc $A'O'B'$, but the remaining arc $2\pi - A'O'B'$, say this is the arc $\overparen{A'B'}$. The integral in regard to u' is thus not the rectilinear integral $(A'B')$, but the integral along the just-mentioned circular arc, say this is denoted by $(\overparen{A'B'})$; and we thus have

$$(AB) = \pm N(\overparen{A'B'}).$$

But we have $(\overparen{A'B'})$ = or not $= (A'B')$, according as the chord $A'B'$ and the arc $\overparen{A'B'}$ do not include between them either of the points C', D', or include between them one or both of these points; and in the same cases respectively

$$(AB) = \text{ or not } = \pm N(A'B').$$

Of course we may in any way interchange the letters, and write under the like circumstances

$$(AC) = \text{ or not } = \pm N(A'C'),\ \&\text{c.}$$

Suppose now that $ABCD$ is a convex quadrilateral, and consider first in regard to (AB), and next in regard to (AC), the three transformations $A'B'C'D' = BADC$, $= CDAB$, and $= DCBA$, respectively. We have here a figure as in the paper "On the circular relation of Möbius," [628], p. 617 of this volume, the points O_1, O_2, O_3 belonging to the three cases respectively. It will be observed in the figure, and it is easy to see generally, that the points O_1 and O_3 are interior, the point O_2 exterior. We have $N = 1$, and therefore

$$(AB) = \text{or not} = \pm(AB), = \text{or not} = \pm(CD), = \text{or not} = \pm(CD),$$

according as

(1) the chord AB and the arc $\smile{AB}$ of ABO_1 do not or do inclose C and D or either of them;

(2) the chord CD and the arc $\smile{CD}$ of CDO_2 do not or do inclose A and B or either of them;

(3) the chord CD and the arc $\smile{CD}$ of CDO_3 do not or do inclose A and B or either of them.

The first test gives merely the identity $(AB) = \pm(AB)$; the other two each of them give $(AB) = \pm(CD)$, as is seen from the positions of the points O_1, O_2, O_3.

Next, apply the test to AC; we have

$$(AC) = \text{or not} = \pm(BD), = \text{or not} = \pm(AC), = \text{or not} = \pm(BD),$$

according as

(1) the chord AC and the arc $\smile{AC}$ of ACO_1 do not or do inclose B and D or either of them;

(2) the chord BD and the arc $\smile{BD}$ of BDO_2 do not or do inclose A and C or either of them;

(3) the chord BD and the arc $\smile{BD}$ of BDO_3 do not or do inclose A and C or either of them.

In the second case, neither A nor C is inclosed, but we have merely the identity $(AC) = \pm(AC)$; in the first case, B is inclosed and, in the third case, C is inclosed; and the tests each give (AC) not $= \pm(BD)$.

I have not taken the trouble of drawing the figure for a reentrant quadrilateral $ABCD$; the mere symmetry is here enough to show that, having one, we have all three, of the relations in question

$$(AD) = \pm(BC), \quad (BD) = \pm(CA), \quad (CD) = \pm(AB).$$

END OF VOL. IX.