33.

ON HAMILTON'S QUADRATIC EQUATION AND THE GENERAL UNILATERAL EQUATION IN MATRICES.

[*Philosophical Magazine*, XVIII. (1884), pp. 454—458.]

IN the *Philosophical Magazine* of May last I gave a purely algebraical method of solving Hamilton's equation in Quaternions, but did not carry out the calculations to the full extent that I have since found is desirable. The completed solution presents some such very beautiful features, that I think no apology will be required for occupying a short space of the *Magazine* with a succinct account of it.

Hamilton was led to this equation as a means of calculating a continued fraction in quaternions, and there is every reason for believing that the Gaussian theory of Quadratic Forms in the theory of numbers may be extended to quaternions or binary matrices, in which case the properties of the equation with which I am about to deal will form an essential part of such extended theory*. Let us take a form slightly more general than that before considered, namely, the form

$$px^2 + qx + r = 0,$$

with the understanding that the determinant of p (if we are dealing with matrices), or its tensor if with quaternions, differs from zero. Let us construct the ternary quadratic

$$au^2 + 2buv + 2cuw + dv^2 + 2evw + fw^2,$$

defined as the determinant of $up + vq + wr$, on the one supposition, or by means of the equations

$$a = Tp^2, \quad d = Tq^2, \quad f = Tr^2, \quad b = SpSq - SVpVq,$$

$$c = SpSr - SVpVr, \quad e = SqSr - SVqVr,$$

on the other supposition.

* I have found, and stated, I believe, in the form of a question in the *Educational Times* some years ago, that any fraction whose terms are real integer quaternions may be expressed as a finite continued fraction, the greatest-common-measure process being applicable to its two terms, provided *both* their Moduli are not odd multiples of an odd power of 2, which can always be guarded against by a previous preparation of the fraction.

On referring to the article of May [p. 226 above], it will be seen that the solution of the equation may be made to depend on the roots of a cubic equation in the quantity therein called λ. When fully worked out, this equation will be found to take the remarkable form $e^{\lambda\Omega}.I=0$, where I is the invariant of the ternary quadratic above written, and $\Omega = 2a\delta_c - a\delta_d$. It may also be shown that

$$x = -\frac{(p+b-u)(q-c-v)}{2\lambda},$$

where u is a two-valued function of λ, and v a linear function of u.

I shall suppose that I, the final term in the equation in λ, differs from zero: the solution of the given equation in x will then be what may be termed *regular*, and will consist of three pairs of actual and determinate roots. When $I=0$, the solution ceases to be regular; some of the roots may disappear from the sphere of actuality, or may remain actual but become indeterminate, or these two states of things may coexist. The first coefficient of the equation in λ is a, the determinant of p (or its squared tensor), which also must not be zero, as in that case one root at least of λ would be infinite. Let us suppose, then, that neither a nor I vanishes. The very interesting question presents itself as to what kind of equalities can arise among the *three* pairs of roots, and what are the conditions of such arising.

This equation admits of an extremely interesting and succinct answer as follows:—Let m represent $\frac{c+2d}{3}$; the equalities between the roots of the given equation in x will be completely governed, and are definable by the equalities existing between those of the biquadratic binary form

$$(a, b, m, e, f)(X, Y)^4 \text{*}.$$

* If the equation is regarded as one in quaternions, the determining biquadratic is the modulus of x^2+xp+q; from which it follows immediately that, if p, q are *real* quaternions, all the four roots, say α, β, γ, δ, are imaginary. It may be shown that the roots of Hamilton's determining cubic are

$$d-\frac{(\alpha+\beta)(\gamma+\delta)}{4},\quad d-\frac{(\alpha+\gamma)(\beta+\delta)}{4},\quad d-\frac{(\alpha+\delta)(\beta+\gamma)}{4},$$

and these therefore are (as shown also by Hamilton) all of them real. The biquadratic serves to determine the points in which the variable conic associated to the equation px^2+qx+r (that is, the determinant to $xp+yq+zr$) is intersected by the absolute conic $xz-y^2$. Each root of the given equation corresponds to a side of the complete quadrilateral formed by the four points of intersection of these two conics; and thus we see that there are five cases to consider when the variable conic is a conic proper, according as it intersects or touches the fixed conic (which can happen in four different ways); and seven other cases where the conic degenerates into two intersecting or two coincident lines (in which cases the solution becomes irregular); namely, the intersecting lines may cut or touch in one or two points the fixed one, and may cut or touch the conic at their point of intersection, which gives five cases; and the coincident lines may cut or touch the fixed conic, which gives two more. Hence there are in all twelve principal cases to consider in Hamilton's form of the Quadratic Equation in Quaternions: or rather thirteen, for the case of the variable and fixed conics coinciding must not be lost sight of.

If the biquadratic has two equal roots, the given quadratic will have two pairs of equal roots.

If the biquadratic has two pairs of equal roots, the given quadratic will have four equal roots.

If the biquadratic has three equal roots, the quadratic will have three pairs of equal roots.

If the biquadratic has all its roots equal, the quadratic will have all its roots equal.

In the first case two of the three pairs of roots of the given quadratic coincide, or merge into a single pair.

In the second case, not only two pairs merge into one pair, but the two roots of that pair coincide with one another.

In the third case the three pairs merge into a single pair.

In the fourth case the two members of that single pair coincide with one another.

So long as the equation in x remains regular, no kind of equalities can exist between the roots other than those above specified.

For instance, let us consider the possibility of two values of x, and no more, becoming equal. First, let us inquire what is the condition to be satisfied in order that the scalar parts of two roots which belong to the same pair shall become equal. It may be shown that the sufficient and necessary condition that this may take place is that the irreducible sub-invariant of degree 3 and weight 6 (that is, the first coefficient of the irreducible skew-covariant of the associated biquadratic form $[a, b, m, e, f]$) shall vanish.

If, now, the *vectors* as well as the *scalars* of the two roots are to be equal, it may be shown that the *second* as well as the first coefficient of the skew-covariant must vanish. But this cannot happen without the discriminant vanishing*; for it may easily be seen that the discriminant of a binary biquadratic with its sign changed is equal to sixteen times the product of the first and last coefficients, less the product of the second and penultimate coefficients of its irreducible skew-covariant. Hence when two roots belonging to the same pair of the given quadratic coincide, two values of λ become equal, and therefore all four roots belonging to two pairs merge into one.

Again, it is not possible for two roots belonging to two pairs corresponding to two different values of λ to coincide; for in such case the expression

* The first two coefficients of the skew-covariant vanishing implies the existence of two pairs of equal roots and *vice versâ*. This is on the supposition made that a, the first coefficient of the given quartic, is not zero.

given for x shows that $pq, p, q, 1$ would be connected by a linear equation. But when this happens (as has been shown by me elsewhere), the invariant of the associated ternary quartic vanishes and the equation ceases to be regular. Thus, then, it appears that it is impossible for a single relation of equality (*and no more*) to exist between the roots of the given equation when its form is regular. So, again, it may be shown that it is impossible for four, and no more, relations of equality to exist between the roots.

It need hardly be added, that the equation $px^2 + qx + r = 0$ ceases to be regular when q or r vanishes.

The reader may satisfy himself as to the truth of what has been alleged as to the relation of the discriminant of a binary biquadratic to the coefficients of its skew-covariant by simple verification of the identity

$$\begin{aligned} &16\,(a^2d - 3abc + 2b^3)\,(e^2b - 3edc + 2d^3) \\ &\quad - (a^2e + 2abd - 9c^2a + 6b^2c)\,(e^2a + 2edb - 9ec^2 + 6d^2c) \\ &\quad = 27\,(ace + 2bcd - c^3 - b^2e - ad^2)^2 - (ae - 4bd + 3c^2)^3. \end{aligned}$$

The biquadratic equation in X, Y is what the determinant of $\lambda p + \mu q + \nu r$ becomes when X^2, XY, Y^2 are substituted therein for λ, μ, ν; so that we may say that $(a, b, m, e, f)(x, 1)^4$ is the determinant of $px^2 + qx + r$, when x is regarded as an ordinary quantity. Let ϕx be any quadratic factor of this biquadratic function in x: I have found that $\phi x = 0$ will be the *identical* equation to one of the roots of the given equation $fx = 0$, where

$$fx = px^2 + qx + r.$$

Between the two equations $fx = 0$, $\phi x = 0$, x^2 may be eliminated and x found in terms of known quantities: ϕx will have six different values, which will give the six roots of $fx = 0$. It is far from improbable that a similar solution applies to a unilateral equation $fx = 0$ of any degree n in matrices of any order ω.

Call Fx the determinant of fx when x is regarded as an ordinary quantity; then, if ϕx is an algebraical factor of the degree ω in x contained in Fx, it would seem to be in all probability true that $\phi x = 0$ is the identical equation to one of the roots of $fx = 0$; and, *vice versâ*, that the function identically zero of any such root is a factor of Fx. By combining the equations $fx = 0$, $\phi x = 0$, all the powers of x except the first may be eliminated, and thus every root of x determined. The solution of the given equation will depend upon the solution of an ordinary equation of the degree $n\omega$, and the number of roots will be the number of ways of combining $n\omega$ things ω and ω together. Thus, for a cubic equation in quaternions the number of roots would be $\frac{1}{2}6 \cdot 5$, or 15. In the May number of this *Magazine* [p. 229 above] it was supposed to be shown to be 21; but it is quite conceivable that this determination may be erroneous, especially as it was deduced from general considerations of the degrees of a certain system of equations without attention being paid to their particular form, which might very well be such as to occasion a fall in the *order* of the system. I am strongly inclined, with the new light I have gained on the subject, to believe that such must be the case, and that the true number of roots for a unilateral equation in quaternions of the degree n is $2n^2-n$*; in which case the theorem above stated, and which may be viewed as a marvellous generalization of the already marvellous Hamilton-Cayley Theorem of the identical equation, will be undoubtedly true for all values of n and ω. But I can only assert positively at present that it is true for the case of $n=1$ whatever ω may be, and for the case of $n=2$, $\omega=2$†.

* From the number 21 above referred to, now known to be erroneous, the general value was inferred to be n^3-n^2+n, whereas it is demonstrably $2n^2-n$ only for the *general* unilateral equation of degree n in quaternions, as I proved it to be for the *Jerrardian* form of that equation.

† I have since obtained an easy proof of the truth of the conjectural theorem for all values of n and ω; see the *Comptes Rendus* of the Institute of France for October 20th last [p. 197 above].