35.

ON THE D'ALEMBERT-CARNOT GEOMETRICAL PARADOX AND ITS RESOLUTION.

[*Messenger of Mathematics,* XIV. (1885), pp. 92—96.]

I WILL presently state the simple geometrical problem which led D'Alembert to call into question the validity of the received Cartesian doctrine of positive and negative geometrical magnitudes, and which, according to Carnot, furnishes an unanswerable argument against it. See Mouchot, *La réforme Cartésienne*, pp. 74, 75.

Against this doctrine, presented in its crude form, the objections of these illustrious impugners of it are unquestionably well founded and unanswerable; but the inference to be drawn from this is not that no such or such-like doctrine reposing on an unassailable logical basis exists or is capable of being established (woe worth the day! when such a conclusion should be admitted), but that the doctrine as usually stated is incomplete and requires a supplement.

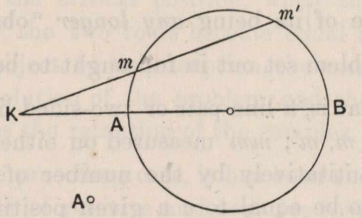
This has been anticipatively furnished by me many years ago in this very *Journal*, and in conjunction with the substitution of positive and negative indefinite rotation in lieu of Euclid's positive and limited angular magnitude, made the basis of a strictly logical deduction (which was before wanting) of the trigonometrical canon.

It consists in the notion of a line having, so to say, sides (returning upon itself at its two semi-points at infinity), or to put the matter in a more practical form, in regarding an Euclidean indefinite straight line as representing two distinct lines locally coincident, but running in contrary directions, and in referring the algebraical sign of any rectilinear segment to the concurrence or discordance of its flow (which is represented by the order in which its two extremities are named or written down) with that of the indefinite line, upon which it is supposed to be carried.

Thus, for example, AB taken on the upper side of a line or line-pair will be the negative of AB taken on the same side, but the same as BA taken on the under side.

I will now state the D'Alembert-Carnot problem. "Voici" says Carnot, "un exemple aussi simple que frappant, qui seul suffit pour renverser toute cette doctrine" of positive and negative magnitudes.

"D'un point K, pris hors d'un cercle donné, soit proposé de mener une droite Kmm', telle que la portion mm', interceptée dans le cercle, soit égale à une droite donnée.



"Du point K, et par le centre du cercle menons une droite KAB qui rencontre la circonférence en A et B. Supposons $KA = a$, $KB = b$, $mm' = c$, $Km = x$. On aura donc par les propriétés du cercle

$$ab = x(c + x) = cx + x^2$$

donc
$$x^2 + cx - ab = 0$$

ou
$$x = -\tfrac{1}{2}c \pm \sqrt{(\tfrac{1}{4}c^2 + ab)}.$$

x a deux valeurs: la première, qui est positive, satisfait sans difficulté à la question; mais que signifie la seconde, qui est négative? Il paraît qu'elle ne peut répondre qu'au point m', qui est le second de ceux où Km coupe la circonférence; et, en effet, si l'on cherche directement Km', en prenant cette droite pour l'inconnue x, on aura

$$x(x - c) = ab$$

ou
$$x = \tfrac{1}{2}c \pm \sqrt{(\tfrac{1}{4}c^2 + ab)}$$

dont la valeur positive est précisément la même que celle qui s'était présentée dans le premier cas avec le signe négatif. Donc, quoique les deux racines de l'équation

$$x = -\tfrac{1}{2}c \pm \sqrt{(\tfrac{1}{4}c^2 + ab)}$$

soient l'une positive et l'autre négative, elles doivent être prises toutes les deux dans le même sens par rapport au point fixe K. Ainsi, la règle qui veut que ces racines soient prises en sens opposés porte à faux. Si au contraire le point fixe K était pris sur le diamètre même AB et non sur le prolongement, on trouverait pour x deux valeurs positives et cependant elles devraient être prises en sens contraires l'une de l'autre. La règle est donc encore fausse pour ce cas.

"Si l'on dit que ce n'est pas ainsi qu'il faut entendre ce principe, que les racines positives et négatives doivent être prises en sens opposés, je demanderai comment il faut l'entendre? et j'en conclurai par là même qu'il faut une explication pour empêcher qu'il ne soit pris dans l'acceptation la plus naturelle. Il suit que ce principe est obscur et vague."

The answer has been already given to the question, "comment il faut entendre ce principe," and it will be seen in such a way as to remove all grounds for the charge of its being *any longer* "obscur et vague."

This is how the problem set out in full ought to be enunciated:

A complete line (that is, a line-pair or two-sided line) drawn from K cuts the circle in the points m, m'; mm' measured on either side of the line (and of course denoted quantitatively by the number of units of given length which it contains) is to be equal to c a given positive or negative number. Required the value of Am.

(1) Suppose K to be exterior to the circle as in the diagram above.

I distinguish the two sides of the complete line, as the under and upper line, and suppose the flow of the under one to make an acute Euclidean angle with the flow from K to the centre of the circle. In all cases

$$Km' = Km + mm',$$

and consequently the equation for finding x remains always $x^2 + cx = ab$, of which the two roots are $-\frac{1}{2}c + \sqrt{(\frac{1}{4}c^2 + ab)}$ and $-\frac{1}{2}c - \sqrt{(\frac{1}{4}c^2 + ab)}$.

Adhering to the letters of the diagram, if c is positive the two values of x will correspond to Am on the under line and Am' on the upper line of the line-pair. If, again, c is negative, the two values of x will correspond to Am on the upper and Am' on the under one.

(2) Suppose K to be within the circle.

It will still be true (paying attention to the signs) that $Km' = Km + mm'$ (that being a universal identity in algebraical geometry), but the algebraical values of KA, KB being contrary, we may regard KA as positive and equal to a, KB as negative and equal to $-b$, and shall have the equation

$$x^2 + cx = -ab,$$

of which the two roots are

$$-\tfrac{1}{2}c + \sqrt{(\tfrac{1}{4}c^2 - ab)}, \quad -\tfrac{1}{2}c - \sqrt{(\tfrac{1}{4}c^2 - ab)}.$$

Understand by the *two segments* Km and Km'.

We may suppose the indefinite line-pair mKm' to swing round K, its under-side in the position of coincidence with the diameter having the same flow as KA; then, if c is positive, until the swinging line revolving with the sun has described a right angle, the first root will be the *infrà*-diametral segment taken on the lower line (or side), and the second root the *suprà*-diametral segment taken on the upper line (or side) of the line-pair (or complete line); in the next quadrant of rotation the first root will be the *suprà*-diametral segment on the under and the second root the *infrà*-diametral segment on the upper side of the complete line. When c is negative a similar statement may be made if only the words *under* and *upper* are interchanged. In the critical position, when the swinger is at right angles to the diameter, the two roots become equal and undistinguishable; but throughout and subject to no exception, the complex of the two roots contains the complete solution of the problem, and the complete solution of the problem necessitates the retention of the complex of the two roots.

Thus, then, as in the preceding case, it has been shown that the Cartesian view of the equipollence of positive and negative roots (the latter Descartes influenced by hereditary prepossessions calls *radices falsae*) is made exact through the intermediation of the conception of sides to a line. D'Alembert and Carnot are entitled to the gratitude of Geometers and all lovers of truth for raising objections so perfectly well founded to the then, and even now, too prevalent interpretation of the meaning of the geometrical positive and negative, but the difficulty which they so justly appreciated and so clearly expressed is overcome and exists no longer.

P.S. I am informed that M. Laguerre has emitted the same view as that I have set forth relative to the sign to be given to geometrical distances, and made use of the same conception of the double or complete line-carrier.

My note on the subject appeared before my exodus across the Atlantic, probably nine or ten years ago. M. Laguerre's publication must have been many years posterior to this. The references to the reappearance of the theory on the other side of the Channel, obligingly furnished to me by M. Mannheim in Paris, have unfortunately got mislaid. I believe the communication containing it was made by M. Laguerre within the last three or four years, but it has already had time to find its way into some of the most esteemed French text-books. Being not only true but *the* truth, it must eventually find universal acceptance. It is not without interest (it seems to me) that we may regard a double or complete right line as a sort of embryonic embodiment of the idea of a Riemann Surface.