38.

ON RECIPROCANTS.

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IN a note on Invariant Derivatives in the September number of the *Messenger* I have given a definition and examples of reciprocants.

If in any of the forms at the end of the postscript to the note we restore to $a, b, c, \dots$ their values $\delta_x y, \delta_x^2 y, \delta_x^3 y, \dots$ any such function divided by a certain power of $\delta_x y$ will change its sign, but otherwise remain unaltered when x and y are interchanged. The index of that power is the degree added to half the weight and will be called the index of the reciprocant. Any product of i of such reciprocants will be a reciprocant of the same kind or contrary kind to those in the table (subsequent to a) according as i is odd or even. In the latter case the interchange of x and y will leave the function absolutely unaltered. Reciprocants which cause a change of sign will be said to be of an odd, those which cause no change of sign of an even character. Any linear function of reciprocants of the same weight, degree, and *character* will be itself a reciprocant of that character, but reciprocants of opposite characters cannot be combined to form a new reciprocant: those of an odd character may be regarded as analogous to skew, those of an even character to non-skew seminvariants; the rule against combining forms of opposite characters becomes superfluous in the case of seminvariants, because those that offer themselves for combination as having the same weight and degree must of necessity be of like character. Any reciprocant being given there is a simple *ex post facto* rule for assigning its character without any knowledge of the mode of its genesis, namely its character is odd or even according as the smallest number of letters other than a in any of its terms is odd or even. Thus the *character* of a reciprocant whose leading term is a^2e, or ab^2e, or $abce$ is odd; that of one whose *leading* term is abe or abf is even, as is also that of the remarkable reciprocant $bd-5c^2$ in which no power of a appears.

A further important distinction between the two theories* is that there are two linear reciprocants a and b but only one linear seminvariant. As an illustration of the combinatorial *law* of *like character* it will be seen that if we operate upon $2ac-3b^2$ with the operator

$$a(b\delta a + c\delta b) - 3b,$$

* That is of reciprocants and invariants.

we obtain a new reciprocant

$$2ad - 10bc + 9b^3,$$

of which the character is the same as that of b^3, namely both are odd; we may therefore add $-9b^3$ to the latter expression, and then dividing out by $2a$ there results the reciprocant $ad - 5bc$, but we cannot combine $2ac - 3b^2$ with b^2 because these two reciprocants are of opposite characters.

Again, remembering that a is of an even and b of an odd character, the three reciprocants

$$-\tfrac{45}{4}b^4, \quad 5(ac - \tfrac{3}{2}b^2)^2, \quad 3ab(ad - 5bc)$$

are all of an even order, hence we may add them together and divide the sum by a^2, which gives the new reciprocant $3bd - 5c^2$ a form not containing the first letter a.

No seminvariant exists, nor, except the one just given $bd - 5c^2$, have I been able to discover any other reciprocant in which the first letter does not make its appearance†.

The infinite progression of odd reciprocants with the leading terms

$$ac, \quad ad, \quad a.a.e, \quad a.a.f, \quad a.a.a.g, \quad a.a.a.h, \quad \ldots$$

will easily be seen to exist by virtue of the general theorem that any reciprocant of degree, extent, and weight (say briefly of *dew* i, j, w) gives birth to two others of the same character as its own, one of *dew* $i+1, j+1, w+2$, the other of *dew* $i+1, j+2, w+3$.

For let $\qquad \tfrac{1}{2}w + i = \lambda,$

then denoting the operator

$$b\delta_a + c\delta_b + \ldots \text{ by } \Omega,$$

and the result of the action of Ω upon itself $(\Omega *)^2$, which is in fact $\Omega^2 + \Omega_2$ (Ω_2 meaning $c\delta_a + d\delta_b + \ldots$); $(a\Omega - \lambda b) R$ will obviously be a reciprocant of *dew* $i+1, j+1, w+2$, and will give rise to a second reciprocant

$$\{a\Omega - (\lambda + \tfrac{3}{2}) b\} (a\Omega - \lambda b) R,$$

which is $\qquad a^2 (\Omega *)^2 - (2\lambda + \tfrac{1}{2}) ab\Omega R - \lambda acR + (\lambda^2 + \tfrac{3}{2}\lambda) b^2R;$

the last term of this being a reciprocant of the same character as the entire expression may be omitted, and dividing out the residue by a we obtain the second new reciprocant

$$\{a(\Omega *)^2 - (2\lambda + \tfrac{1}{2}) b - \lambda c\} R,$$

which will be of *dew* $i+1, j+2, w+3$, as was to be shown.

It is easy to see that every reciprocant must be a rational integral function of the forms above stated commencing with $a, b, 2ac - 3b^2$ (whose *dew*'s are alternately of the form $i, 2i-1, 3i-2$; $i, i-2, 3i-1$) divided by some power of a. For if any reciprocant contains only the letters $a, b, \ldots$

† Since the above went to press I have made the capital discovery that there are an infinite number of such reciprocants, and that all those of a given weight, extent and degree may be obtained by aid of a certain quadratico-linear partial differential equation.

h, k, l, it may be expressed as a rational integral function of the protomorph in which l first appears and of the letters $a, b, \ldots k$ divided by a power of a, and consequently the reciprocant may be so expressed, and continually repeating this process of substitution it follows that the reciprocant will be a rational integer of the protomorphs exclusively divided by a power of a*: this of course will necessarily be found only to contain combinations of like character; we already know the converse that the sum of all combinations of like character of the protomorphs is a reciprocant†. If any homogeneous reciprocant consists of portions of unlike degree (although of the same index) it is obvious that each portion must be itself a reciprocant, for if $P, P', P'' \ldots$ be such portions, $P + P' + P'' \ldots$ must be identical with $\Pi + \Pi' + \Pi'' + \ldots$ when $\Pi, \Pi', \Pi'' \ldots$ are the same functions of $\alpha, \beta, \gamma \ldots$ (that is, $\delta_y x, \delta_y{}^2 x, \delta_y{}^3 x \ldots$) that $P, P', P'' \ldots$ are of $a, b, c \ldots$. If then we make

$$P - a^{2\lambda}\Pi = \Delta, \quad P' - a^{2\lambda}\Pi' = \Delta' \ldots,$$

we have $\Delta + \Delta' + \Delta'' + \ldots$ identically zero.

But $P, P' \ldots$ being of the same index but different degrees must be of different weights, and consequently $\Delta, \Delta', \ldots$ are of different weights. Hence we must have $\Delta = 0$, $\Delta' = 0$, &c., as was to be shown.

It follows from this that every reciprocating function whatever may be obtained by an algebraical combination of the protomorphs, and consequently by an algebraical combination of the forms

$$\left(\frac{1}{y'^{\frac{1}{2}}}\delta_x\right)^i \log y',$$

* The proof that every seminvariant is a rational integral function of the protomorphs is very similar: any proposed seminvariant is by the method employed in the text shown to be at worst a function of the protomorphs and of b; but the terms involving any power of b must disappear because no identical equation can connect seminvariants with a non-seminvariant b. In the text we see in like manner that any given reciprocant may be reduced to the form $H+K$, where H and K are protomorphic combinations of opposite character, so that one of them will disappear.

† Another general mode of generating a class of reciprocants would be to express any function of $a, b, c, \ldots$ say $\phi(a, b, c, \ldots)$ under the form $\psi(\alpha, \beta, \gamma, \ldots)$. The product $\phi(a, b, c, \ldots)\psi(a, b, c, \ldots)$, or its numerator, will then obviously be a reciprocant. To take a simple example,

$$c = \frac{d^3y}{dx^3} = -\frac{\frac{dx}{dy}\cdot\frac{d^3x}{dy^3} - 3\left(\frac{d^2x}{dy^2}\right)^2}{\left(\frac{dx}{dy}\right)^5} = -\alpha\gamma + 3\beta^2 \div \alpha^5.$$

Hence, by the rule laid down, $c(ac - 3b^2)$, that is, $ac^2 - 3b^2c$ ought to be a reciprocant, which is right, for it is equal to $(2ac - 3b^2)^2 - 9b^4$ divided by a multiple of a. The law that the factors of seminvariants must be seminvariants cannot be extended to the theory of reciprocants. In this case the factors may some or none of them be reciprocants, and the others on reciprocation exchange forms monocyclically or polycyclically with one another. *I add the remark that this is not true of pure reciprocants, that is, those in which $\frac{dy}{dx}$ does not appear. Every factor of a pure reciprocant must be itself a reciprocant.*

and that we should gain nothing in generality by operating with successive operators of the form

$$\left(\frac{1}{y'^{\frac{1}{2}}}\,\delta_x\phi_1\cdot\right),\quad \left(\frac{1}{y'^{\frac{1}{2}}}\,\delta_x\phi_2\cdot\right),\ \ldots$$

where $\phi_1, \phi_2, \ldots$ are arbitrary functions of $y' \pm \frac{1}{y'}$, instead of with the simple operator $\frac{1}{y'^{\frac{1}{2}}}\,\delta_x$ continually repeated.

The results of using the more general operators would only amount to algebraical combinations of the results obtained from the simple forms

$$\left(\frac{1}{y'^{\frac{1}{2}}}\,\delta_x\right)^i \log y',$$

where i may take all values from zero to infinity*.

As in the case of seminvariants so also reciprocants would in extent contain only a finite number of ground-forms; but furthermore for reciprocants limited in degree the number of ground-forms will also be finite. Whether reciprocants which are irreducible for a given extent ever cease to be so and become reducible when the order is increased, as is the case with seminvariants, remains to be seen†.

In order to facilitate the verification of the results obtained and to be obtained it may be well to express the successive derivatives of x in regard to y in terms of those of y in regard to x, that is, of $\alpha, \beta, \gamma, \ldots$ in terms of $a, b, c, \ldots$ as shown in the following short table.

$$\begin{array}{ll}
 & \div \\
a = \alpha & \alpha^2, \\
b = -\beta & \alpha^3, \\
c = -\alpha\gamma + 3\beta^2 & \alpha^5, \\
d = -\alpha^2\delta + 10\alpha\beta\gamma - 15\beta^3 & \alpha^7, \\
e = -\alpha^3\epsilon + 15\alpha^2\beta\delta + 10\alpha^2\gamma^2 - 105\alpha\beta^2\gamma + 105\beta^4 & \alpha^9, \\
f = -\alpha^4\zeta + 21\alpha^3\beta\epsilon + 35\alpha^3\gamma\delta - 210\alpha^2\beta^2\delta - 280\alpha^2\beta\gamma^2 + 1260\alpha\beta^3\gamma - 945\beta^5 & \alpha^{11}, \\
\left.\begin{array}{l} g = -\alpha^5\eta + 28\alpha^4\beta\zeta + 56\alpha^4\gamma\epsilon + 35\alpha^4\delta^2 - 378\alpha^3\beta^2\epsilon - 1260\alpha^3\beta\gamma\delta + 3150\alpha^2\beta^3\delta \\ \qquad - 280\alpha^3\gamma^3 + 6300\alpha^2\beta^2\gamma^2 - 17325\alpha\beta^4\gamma + 10395\beta^6 \end{array}\right\} & \alpha^{13},
\end{array}$$

where $a, b, c, d, e, f, \ldots$ represent the successive derivatives of y with respect to x; and $\alpha, \beta, \gamma, \delta, \epsilon, \zeta, \ldots$ of x with respect to y.

In any subsequent paper on reciprocants in this Journal, I shall make the absolutely necessary transliteration referred to in a preceding footnote, replacing the present letters $a, b, c, d, \ldots$ by the letters $t, a, b, c, \ldots$ or possibly, for reasons which carry great weight, by the expressions

$$t,\quad 2a,\quad 2.3b,\quad 2.3.4c, \ldots$$

* This is not true of homogeneous reciprocants.

† I have since found that this *is* true for reciprocants, as for seminvariants.