103.

ON CERTAIN DEFINITE INTEGRALS.

[From the *Cambridge and Dublin Mathematical Journal*, vol. VI. (1851), pp. 136—140.]

SUPPOSE that for any positive or negative integral value of r, we have $\psi(x+ra) = U_r \psi x$, U_r being in general a function of x, and consider the definite integral

$$I = \int_{-\infty}^{\infty} \psi x \Psi x \, dx;$$

Ψx being any other function of x. In case of either of the functions ψx, Ψx becoming infinite for any real value α of x, the principal value of the integral is to be taken, that is, I is to be considered as the limit of

$$\left(\int_{\alpha+\epsilon}^{\infty} + \int_{-\infty}^{\alpha-\epsilon}\right) \psi x \, \Psi x \, dx, \quad (\epsilon = 0),$$

and similarly, when one of the functions becomes infinite for several of such values of x.

We have

$$I = \left(\ldots \int_{ra}^{(r+1)a} + \ldots\right) \psi x \Psi x \, dx;$$

or changing the variables in the different integrals so as to make the limits of each a, 0, we have

$$I = \int_0^a \left[\Sigma \, \psi(x+ra) \, \Psi(x+ra)\right] dx,$$

Σ extending to all positive or negative integer values of r, that is,

$$I = \int_0^a \psi x \left[\Sigma \, U_r \Psi(x+ra)\right] dx, \quad \text{.............................(A)}$$

which is true, even when the quantity under the integral sign becomes infinite for particular values of x, provided the integral be replaced by its principal value, that is, provided it be considered as the limit of

$$\left(\int_{\alpha+\epsilon}^{a} + \int_{0}^{\alpha-\epsilon}\right) \psi x \left[\Sigma U_r \Psi (x + ra)\right] dx,$$

or

$$\int_{\epsilon}^{a-\epsilon} \psi x \left[\Sigma U_r \Psi (x + ra)\right] dx;$$

where α, or one of the limiting values a, 0, is the value of x, for which the quantity under the integral sign becomes infinite, and ϵ is ultimately evanescent.

In particular, taking for simplicity $a = \pi$, suppose

$$\psi (x + \pi) = \pm \psi x, \quad \text{or} \quad \psi (x + r\pi) = (\pm)^r \psi x;$$

then observing the equation

$$\Sigma \frac{(\pm)^r 1}{x + r\pi} = \cot x, \quad \text{or} = \operatorname{cosec} x,$$

according as the upper or under sign is taken, and assuming $\Psi x = x^{-\mu}$, we have finally

$$\int_{-\infty}^{\infty} \frac{\psi x\, dx}{x^{\mu}} = \frac{(-)^{\mu-1}}{\Gamma\mu} \int_0^{\pi} \psi x \left[\left(\frac{d}{dx}\right)^{\mu-1} \cot x\right] dx,$$

$$\int_{-\infty}^{\infty} \frac{\psi x\, dx}{x^{\mu}} = \frac{(-)^{\mu-1}}{\Gamma\mu} \int_0^{\pi} \psi x \left[\left(\frac{d}{dx}\right)^{\mu-1} \operatorname{cosec} x\right] dx,$$

the former equation corresponding to the case of $\psi (x + \pi) = \psi x$, the latter to that of $\psi (x + \pi) = - \psi x$.

Suppose $\psi_{,} x = \psi g x$, g being a positive integer. Then

$$\int_{-\infty}^{\infty} \frac{\psi_{,} x\, dx}{x^{\mu}} = g^{\mu-1} \int_{-\infty}^{\infty} \frac{\psi x\, dx}{x^{\mu}};$$

also if $\psi (x + \pi) = \psi x$, then $\psi_{,} (x + \pi) = \psi_{,} x$; but if $\psi (x + \pi) = - \psi x$, then $\psi_{,} (x + \pi) = \pm \psi_{,} x$, the upper or under sign according as g is even or odd. Combining these equations, we have

$\psi (x + \pi) = \psi x$, g even or odd,

$$\int_{-\infty}^{\infty} \frac{\psi g x\, dx}{x^{\mu}} = \frac{(-)^{\mu-1}}{\Gamma(\mu)} g^{\mu-1} \int_0^{\pi} \psi x \left[\left(\frac{d}{dx}\right)^{\mu-1} \cot x\right] dx = \frac{(-)^{\mu-1}}{\Gamma\mu} \int_0^{\pi} \psi g x \left[\left(\frac{d}{dx}\right)^{\mu-1} \cot x\right] dx;$$

$\psi (x + \pi) = - \psi x$, g even,

$$\int_{-\infty}^{\infty} \frac{\psi g x\, dx}{x^{\mu}} = \frac{(-)^{\mu-1}}{\Gamma\mu} g^{\mu-1} \int_0^{\pi} \psi x \left[\left(\frac{d}{dx}\right)^{\mu-1} \operatorname{cosec} x\right] dx = \frac{(-)^{\mu-1}}{\Gamma\mu} \int_0^{\pi} \psi g x \left[\left(\frac{d}{dx}\right)^{\mu-1} \operatorname{cosec} x\right] dx;$$

$\psi (x + \pi) = - \psi x$, g odd,

$$\int_{-\infty}^{\infty} \frac{\psi g x\, dx}{x^{\mu}} = \frac{(-)^{\mu-1}}{\Gamma\mu} g^{\mu-1} \int_0^{\pi} \psi x \left[\left(\frac{d}{dx}\right)^{\mu-1} \operatorname{cosec} x\right] dx = \frac{(-)^{\mu-1}}{\Gamma\mu} \int_0^{\pi} \psi g x \left[\left(\frac{d}{dx}\right)^{\mu-1} \operatorname{cosec} x\right] dx.$$

In particular

$$\int_{-\infty}^{\infty} \frac{\sin x\, dx}{x} = \pi,$$

$$\int_{-\infty}^{\infty} dx\, \frac{\sin x}{x^\mu} = \frac{(-)^{\mu-1}}{\Gamma\mu} \int_0^\pi \sin x \left[\left(\frac{d}{dx}\right)^{\mu-1} \operatorname{cosec} x\right] dx,$$

$$\int_0^\pi \sin gx \left[\left(\frac{d}{dx}\right)^{\mu-1} \cot x\right] dx = g^{\mu-1} \int_0^\pi \sin x \left[\left(\frac{d}{dx}\right)^{\mu-1} \operatorname{cosec} x\right] dx, \quad g \text{ even},$$

$$\int_0^\pi \sin gx \left[\left(\frac{d}{dx}\right)^{\mu-1} \operatorname{cosec} x\right] dx = g^{\mu-1} \int_0^\pi \sin x \left[\left(\frac{d}{dx}\right)^{\mu-1} \operatorname{cosec} x\right] dx, \quad g \text{ odd},$$

$$\int_0^\pi \sin gx \cot x\, dx = \pi, \quad g \text{ even},$$

$$\int_0^\pi \sin gx \operatorname{cosec} x\, dx = \pi, \quad g \text{ odd},$$

$$\int_0^\pi \frac{\tan x\, dx}{x} = 0, \ \&\text{c.},$$

the number of which might be indefinitely extended.

The same principle applies to multiple integrals of any order: thus for double integrals, if $\psi(x+ra,\ y+rb) = U_{r.s}\psi(x,\ y)$, then

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \psi(x,\ y)\,\Psi(x,\ y)\, dx\, dy = \int_0^a\int_0^b \psi(x,\ y)\, \Sigma U_{r,s}\Psi(x+ra,\ y+sb). \ \ldots \text{(B)}$$

In particular, writing w, v for a, b, and assuming $\psi(x+rw,\ y+sv) = (\pm)^r(\pm)^s\psi(x,\ y)$; also $\Psi(x,\ y) = (x+iy)^{-\mu}$, where as usual $i = \sqrt{-1}$,

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\psi(x,\ y)\, dx\, dy}{(x+iy)^\mu} = \frac{(-)^{\mu-1}}{\Gamma\mu}\int_0^w\int_0^v \psi(x,\ y)\left[\left(\frac{d}{dx}\right)^{\mu-1} \Theta(x+iy)\right] dx\, dy, \ldots \text{(B')}$$

where

$$\Theta(x+iy) = \Sigma\, \frac{(\pm)^r(\pm)^s 1}{(x+iy+rw+svi)},$$

Σ extending to all positive or negative integer values of r and s. Employing the notation of a paper in the *Cambridge Mathematical Journal*, "On the Inverse Elliptic Functions," t. IV. [1845], pp. 257—277, [24], we have for the different combinations of the ambiguous sign,

$$1. \quad -,\ -,\quad \Theta(x+iy) = \frac{\mathfrak{G}(x+iy)}{\gamma(x+iy)} = \frac{1}{\phi(x+iy)},$$

$$2. \quad -,\ +,\quad \Theta(x+iy) = \frac{G(x+iy)}{\gamma(x+iy} = \frac{F(x+iy)}{\phi(x+iy)},$$

$$3.\quad +,\ -,\quad \Theta(x+iy)=\frac{g(x+iy)}{\gamma(x+iy)}=\frac{f(x+iy)}{\phi(x+iy)},$$

$$4.\quad +,\ +,\quad \Theta(x+iy)=\frac{\gamma'(x+iy)}{\gamma(x+iy)};$$

where ϕ, f, F are in fact the symbols of the inverse elliptic functions (Abel's notation) corresponding very nearly to sin am, cos am, Δ am. It is remarkable that the last value of Θ cannot be thus expressed, but only by means of the more complicated transcendant γx, corresponding to the $H(x)$ of M. Jacobi. The four cases correspond obviously to

$$\begin{aligned}
&1.\quad \psi(x+rw,\ y+sv)=(-)^{r+s}\,\psi(x,\ y),\\
&2.\quad \psi(x+rw,\ y+sv)=(-)^{r}\,\psi(x,\ y),\\
&3.\quad \psi(x+rw,\ y+sv)=(-)^{s}\,\psi(x,\ y),\\
&4.\quad \psi(x+rw,\ y+sv)=\psi(x,\ y).
\end{aligned}$$

The above formulæ may be all of them modified, as in the case of single integrals, by means of the obvious equation

$$\iint\frac{\psi(gx,\ gy)\,dx\,dy}{(x+iy)^{\mu}}=g^{\mu-2}\iint\frac{\psi(x,\ y)\,dx\,dy}{(x+iy)^{\mu}},\ [\text{limits } \infty,\ -\infty].$$

The most important particular case is

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\phi(x+iy)\,dx\,dy}{(x+iy)}=wv,$$

for in almost all the others, for example in

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\phi(x+iy)\,dx\,dy}{(x+iy)^{\mu}}=\frac{(-)^{\mu-1}}{\Gamma\mu}\int_0^w\int_0^v\phi(x+iy)\left[\left(\frac{d}{dx}\right)^{\mu-1}\frac{1}{\phi(x+iy)}\right]dx\,dy,$$

the second integration cannot be effected.

Suppose next $\psi(x,\ y)$ is one of the functions $\gamma(x+iy)$, $g(x+iy)$, $G(x+iy)$, $\mathfrak{G}(x+iy)$, so that

$$\psi(x+rw,\ y+sv)=(\pm)^r(\pm)^s\,U_{r,s}\psi(x,\ y),$$

where

$$U_{r,s}=(-)^{rs}\,\epsilon^{\beta x(rw-svi)}\,q_{\prime}^{-\frac{1}{2}r^2}\,q^{-\frac{1}{2}s^2},$$

(see memoir quoted). Then, retaining the same value as before of $\Psi(x,\ y)$, we have still the formula (B), in which

$$\Theta(x+iy)=\Sigma\frac{(\pm)^r(\pm)^s\,U_{r,s}}{x+iy+rw+svi}.$$

But this summation has not yet been effected; the difficulty consists in the variable factor $\epsilon^{\beta x(rw-svi)}$ in the numerator, nothing being known I believe of the decomposition of functions into series of this form.

On the subject of the preceding paper may be consulted the following memoirs by Raabe, "Ueber die Summation periodischer Reihen," *Crelle*, t. xv. [1836], pp. 355—364, and "Ueber die Summation harmonisch periodischer Reihen," t. XXIII. [1842], pp. 105—125, and t. xxv. [1843], pp. 160—168. The integrals he considers, are taken between the limits 0, ∞ (instead of $-\infty$, ∞). His results are consequently more general than those given above, but they might be obtained by an analogous method, instead of the much more complicated one adopted by him: thus if $\phi(x+2\pi)=\phi x$, the integral $\int_0^\infty \phi x \frac{dx}{x}$ reduces itself to

$$\Sigma_0^\infty \int_0^{2\pi} \phi x \frac{dx}{x+2r\pi} = \int_0^{2\pi} dx\, \phi x \left[\frac{1}{x} + \Sigma_1{}^\infty \left(\frac{1}{x+2r\pi} - \frac{1}{2r\pi}\right)\right],$$

provided $\int_0^{2\pi} dx \phi x = 0$. The summation in this formula may be effected by means of the function Γ and its differential coefficient, and we have

$$\int_0^\infty \phi x \frac{dx}{x} = -\frac{1}{2\pi}\int_0^{2\pi} dx \frac{\Gamma'\left(\frac{x}{2\pi}\right)}{\Gamma\left(\frac{x}{2\pi}\right)} dx,$$

which is in effect Raabe's formula (10), *Crelle*, t. xxv. p. 166.

By dividing the integral on the right-hand side of the equation into two others whose limits are 0, π, and π, 2π respectively, and writing in the second of these $2\pi - x$ instead of x, then

$$\int_0^\infty \phi x \frac{dx}{x} = -\frac{1}{2\pi}\int_0^\pi \left\{\phi x \frac{\Gamma'\left(\frac{x}{2\pi}\right)}{\Gamma\left(\frac{x}{2\pi}\right)} + \phi(2\pi - x) \frac{\Gamma'\left(1-\frac{x}{2\pi}\right)}{\Gamma\left(1-\frac{x}{2\pi}\right)}\right\} dx;$$

or reducing by

$$\frac{\Gamma'\left(\frac{x}{2\pi}\right)}{\Gamma\left(\frac{x}{2\pi}\right)} = \frac{\Gamma'\left(1-\frac{x}{2\pi}\right)}{\Gamma\left(1-\frac{x}{2\pi}\right)} - \pi \cot \tfrac{1}{2}x,$$

we have

$$\int_0^\infty \phi x \frac{dx}{x} = \tfrac{1}{2}\int_0^\pi \phi x \cot \tfrac{1}{2}x\, dx - \frac{1}{2\pi}\int_0^\pi [\phi x + \phi(2\pi - x)] \frac{\Gamma'\left(1-\frac{x}{2\pi}\right)}{\Gamma\left(1-\frac{x}{2\pi}\right)} dx,$$

which corresponds to Raabe's formula (10′). If $\phi(-x) = -\phi x$, so that $\phi x + \phi(2\pi - x) = 0$, the last formula is simplified; but then the integral on the first side may be replaced by $\frac{1}{2}\int_{-\infty}^{\infty} \phi x \frac{dx}{x}$, so that this belongs to the preceding class of formulæ.