104.

ON THE THEORY OF PERMUTANTS.

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A FORM may by considered as composed of blanks which are to be filled up by inserting in them specializing characters, and a form the blanks of which are so filled up becomes a symbol. We may for brevity speak of the blanks of a symbol in the sense of the blanks of the form from which such symbol is derived. Suppose the characters are 1, 2, 3, 4, ..., the symbol may always be represented in the first instance and without reference to the nature of the form, by $V_{1234}\ldots$ And it will be proper to consider the blanks as having an invariable order to which reference will implicitly be made; thus, in speaking of the characters 2, 1, 3, 4, ... instead of as before 1, 2, 4, ... the symbol will be $V_{2134}\ldots$ instead of $V_{1234}\ldots$. When the form is given we shall have an equation such as

$$V_{1234} = P_{12}\, Q_3\, R_4 \ldots \quad \text{or} \quad = P_{12} P_{34} \ldots \quad \&\text{c.},$$

according to the particular nature of the form.

Consider now the characters 1, 2, 3, 4, ..., and let the primitive arrangement and every arrangement derivable from it by means of an even number of inversions or interchanges of two characters be considered as positive, and the arrangements derived from the primitive arrangement by an odd number of inversions or interchanges of two characters be considered as negative; a rule which may be termed "the rule of signs." The aggregate of the symbols which correspond to every possible arrangement of the characters, giving to each symbol the sign of the arrangement, may be termed a "Permutant;" or, in distinction from the more general functions which will presently be considered, a simple permutant, and may be represented by enclosing the symbol in brackets, thus $(V_{1234}\ldots)$. And by using an expression still more elliptical than the blanks of a symbol, we may speak of the blanks of a permutant, or the characters of a permutant.

As an instance of a simple permutant, we may take

$$(V_{123}) = V_{123} + V_{231} + V_{312} - V_{132} - V_{213} - V_{321};$$

and if in particular $V_{123} = a_1 b_2 c_3$, then

$$(V_{123}) = a_1 b_2 c_3 + a_2 b_3 c_1 + a_3 b_1 c_2 - a_1 b_3 c_2 - a_2 b_1 c_3 - a_3 b_2 c_1.$$

It follows at once that a simple permutant remains unaltered, to the sign *près* according to the rule of signs, by any permutations of the characters entering into the permutant. For instance,

$$(V_{123}) = (V_{231}) = (V_{312}) = -(V_{132}) = -(V_{213}) = -(V_{321}).$$

Consequently also when two or more of the characters are identical, the permutant vanishes, thus

$$V_{113} = 0.$$

The form of the symbol may be such that the symbol remains unaltered, to the sign *près* according to the rule of signs, for any permutations of the characters in certain particular blanks. Such a system of blanks may be termed a quote. Thus, if the first and second blanks are a quote,

$$V_{123} = -V_{213}, \quad V_{132} = -V_{312}, \quad V_{231} = -V_{321},$$

and consequently

$$(V_{123}) = 2\,(V_{123} + V_{231} + V_{312});$$

and if the blanks constitute one single quote,

$$(V_{123}\cdots) = N V_{123}\cdots,$$

where $N = 1.2.3 \ldots n$, n being the number of characters. An important case, which will be noticed in the sequel, is that in which the whole series of blanks divide themselves into quotes, each of them containing the same number of blanks. Thus, if the first and second blanks, and the third and fourth blanks, form quotes respectively,

$$\tfrac{1}{6}(V_{1234}) = V_{1234} + V_{1342} + V_{1423} + V_{3412} + V_{4213} + V_{2314}.$$

It is easy now to pass to the general definition of a "Permutant." We have only to consider the blanks as forming, not as heretofore a single set, but any number of distinct sets, and to consider the characters in each set of blanks as permutable *inter se* and not otherwise, giving to the symbol the sign compounded of the signs corresponding to the arrangements of the characters in the different sets of blanks. Thus, if the first and second blanks form a set, and the third and fourth blanks form a set,

$$((V_{1234})) = V_{1234} - V_{2134} - V_{1243} + V_{2143}.$$

The word 'set' will be used throughout in the above technical sense. The particular mode in which the blanks are divided into sets may be indicated either in words or by some superadded notation. It is clear that the theory of permutants depends ultimately on that of simple permutants; for if in a compound permutant we first write down all the terms which can be obtained, leaving unpermuted the characters of a particular set, and replace each of the terms so obtained by a simple permutant having for its characters the characters of the previously unpermuted set, the result is obviously the original compound permutant. Thus, in the above-mentioned case, where the first and second blanks form a set and the third and fourth blanks form a set,

$$((V_{1234})) = (V_{1234}) - (V_{1243}),$$

or

$$((V_{1234})) = (V_{1234}) - (V_{2134}),$$

in the former of which equations the first and second blanks in each of the permutants on the second side form a set, and in the latter the third and fourth blanks in each of the permutants on the second side form a set, the remaining blanks being simply supernumerary and the characters in them unpermutable. It should be noted that the term quote, as previously defined, is only applicable to a system of blanks belonging to the same set, and it does not appear that anything would be gained by removing this restriction.

The following rule for the expansion of a simple permutant (and which may be at once extended to compound permutants) is obvious. Write down all the distinct terms that can be obtained, on the supposition that the blanks group themselves in any manner into quotes, and replace each of the terms so obtained by a compound permutant having for a distinct set the blanks of each assumed quote; the result is the original simple permutant. Thus in the simple permutant (V_{1234}), supposing for the moment that the first and second blanks form a quote, and that the third and fourth blanks form a quote, this leads to the equation

$$(V_{1234}) = +((V_{1234})) + ((V_{1342})) + ((V_{1423})) + ((V_{3412})) + ((V_{4213})) + ((V_{2314})),$$

where in each of the permutants on the second side the first and second blanks form a set, and also the third and fourth blanks.

The blanks of a simple or compound permutant may of course, without either gain or loss of generality, be considered as having any particular arrangement in space, for instance, in the form of a rectangle: thus $V_{\substack{12\\34}}$ is neither more nor less general than V_{1234}. The idea of some such arrangement naturally presents itself as affording a means of showing in what manner the blanks are grouped into sets. But, considering the blanks as so arranged in a rectangular form, or in lines and columns, suppose in the first instance that this arrangement is independent of the grouping of the blanks into sets, or that the blanks of each set or of any of them are distributed at random in the different lines and columns. Assume that the form is such that a symbol

$$V_{\substack{\alpha\beta\gamma\dots\\ \alpha'\beta'\gamma'\dots\\ \vdots}}$$

is a function of symbols $V_{\alpha\beta\gamma\dots}$, $V_{\alpha'\beta'\gamma'\dots}$, &c. Or, passing over this general case, and the case (of intermediate generality) of the function being a symmetrical function, assume that

$$V_{\substack{\alpha\beta\gamma\dots\\ \alpha'\beta'\gamma'\dots\\ \vdots}}$$

is the product of symbols $V_{\alpha\beta\gamma\ldots}$, $V_{\alpha'\beta'\gamma'\ldots}$, &c. Upon this assumption it becomes important to distinguish the different ways in which the blanks of a set are distributed in the different lines and columns. The cases to be considered are: (*A*). The blanks of a single set or of single sets are situated in more than one column. (*B*). The blanks of each single set are situated in the same column. (*C*). The blanks of each single set form a separate column. The case (*B*) (which includes the case (*C*)) and the case (*C*) merit particular consideration. In fact the case (*B*) is that of the functions which I have, in my memoir on Linear Transformations in the *Journal*, [13, 14] called hyperdeterminants, and the case (*C*) is that of the particular class of hyperdeterminants previously treated of by me in the *Cambridge Philosophical Transactions*, [12] and also particularly noticed in the memoir on Linear Transformations. The functions of the case (*B*) I now propose to call "Intermutants," and those in the case (*C*) "Commutants." Commutants include as a particular case "Determinants," which term will be used in its ordinary signification. The case (*A*) I shall not at present discuss in its generality, but only with the further assumption that the blanks form a single set (this, if nothing further were added, would render the arrangement of the blanks into lines and columns valueless), and moreover that the blanks of each line form a quote: the permutants of this class (from their connexion with the researches of Pfaff on differential equations) I shall term "Pfaffians." And first of commutants, which, as before remarked, include determinants.

The general expression of a commutant is

$$\left(V_{\begin{matrix}11\ldots\\22\\\vdots\\nn\end{matrix}}\right);\quad\text{or}\quad\left(\begin{matrix}11\ldots\\22\\\vdots\\nn\end{matrix}\right)$$

and (stating again for this particular case the general rule for the formation of a permutant) if, permuting the characters in the same column in every possible way, considering these permutations as positive or negative according to the rule of signs, one system be represented by

$$\begin{matrix}r_1s_1\ldots\\r_2s_2\\\vdots\\r_ns_n\end{matrix}$$

the commutant is the sum of all the different terms

$$\pm\pm\ldots V_{r_1s_1}\ldots V_{r_2s_2}\ldots V_{r_ns_n}\ldots$$

The different permutations may be formed as follows: first permute the characters in all the columns except a single column, and in each of the arrangements so obtained permute entire lines of characters. It is obvious that, considering any one of the arrangements obtained by permutations of the characters in all the columns but one, the permutations of entire lines and the addition of the proper sign will only reproduce

3—2

the same symbol—in the case of an even number of columns constantly with the positive sign, but in the case of an odd number of columns with the positive or negative sign, according to the rule of signs. For the inversion or interchange of two entire lines is equivalent to as many inversions or interchanges of two characters as there are characters in a line, that is, as there are columns, and consequently introduces a sign compounded of as many negative signs as there are columns. Hence

THEOREM. A commutant of an even number of columns may be calculated by considering the characters of any one column (no matter which) as supernumerary unpermutable characters, and multiplying the result by the number of permutations of as many things as there are lines in the commutant.

The mark † added to a commutant of an even number of columns will be employed to show that the numerical multiplier is to be omitted. The same mark placed over any one of the columns of the commutant will show that the characters of that particular column are considered as non-permutable.

A determinant is consequently represented indifferently by the notations

$$\begin{pmatrix} 11 \\ 22 \\ \vdots \\ nn \end{pmatrix}^{\dagger}, \quad \begin{pmatrix} \overset{\dagger}{1}1 \\ 22 \\ \vdots \\ nn \end{pmatrix}, \quad \begin{pmatrix} 1\overset{\dagger}{1} \\ 22 \\ \vdots \\ nn \end{pmatrix};$$

and a commutant of an odd number of columns vanishes identically.

By considering, however, a commutant of an odd number of columns, having the characters of some one column non-permutable, we obtain what will in the sequel be spoken of as commutants of an odd number of columns. This non-permutability will be denoted, as before, by means of the mark † placed over the column in question, and it is to be noticed that it is not, as in the case of a commutant of an even number of columns, indifferent over which of the columns the mark in question is placed; and consequently there would be no meaning in simply adding the mark † to a commutant of an odd number of columns.

A commutant is said to be symmetrical when the symbols $V_{\alpha\beta\gamma\ldots}$ are such as to remain unaltered by any permutations *inter se* of the characters $\alpha, \beta, \gamma \ldots$ A commutant is said to be skew when each symbol $V_{\alpha\beta\gamma\ldots}$ is such as to be altered in sign only according to the rule of signs for any permutations *inter se* of the characters $\alpha, \beta, \gamma \ldots$, this of course implies that the symbol $V_{\alpha\beta\gamma\ldots}$ vanishes when any two of the characters $\alpha, \beta, \gamma \ldots$ are identical. The commutant is said to be demi-skew when $V_{\alpha, \beta, \gamma \ldots}$ is altered in sign only, according to the rule of signs for any permutation *inter se* of non-identical characters $\alpha, \beta, \gamma, \ldots$

An intermutant is represented by a notation similar to that of a commutant. The sets are to be distinguished, whenever it is possible to do so, by placing in contiguity the symbols of the same set, and separating them by a stroke or bar from the symbols of the adjacent sets. If, however, the symbols of the same set cannot be placed contiguously, we may distinguish the symbols of a set by annexing to them some auxiliary character by way of suffix or otherwise, these auxiliary symbols being omitted in the final result. Thus

$$\begin{pmatrix} 1 & 1 & 1a \\ \underline{2} & 2 & 2b \\ 3 & \underline{3} & 5a \\ 4 & 3 & 6b \end{pmatrix}$$

would show that 1, 2 of the first column and the 3, 4 of the same column, the 1, 2 and the upper 3 of the second column, and the lower 3 of the same column, the 1, 5 of the third column, and the 2, 6 of the same column, form so many distinct sets,—the intermutant containing therefore

$$(2 \,.\, 2 \,.\, 6 \,.\, 1 \,.\, 2 \,.\, 2 =)\ 96 \text{ terms.}$$

A commutant of an even number of columns may be considered as an intermutant such that the characters of some one (no matter which) of its columns form each of them by itself a distinct set, and in like manner a commutant of an odd number of columns may be considered as an intermutant such that the characters of some one determinate column form each of them by itself a distinct set.

The distinction of symmetrical, skew and demi-skew applies obviously as well to intermutants as to commutants. The theory of skew intermutants and skew commutants has a connexion with that of Pfaffians.

Suppose $V_{\alpha\beta\gamma\ldots} = V_{\alpha+\beta+\gamma\ldots}$ (which implies the symmetry of the intermutant or commutant) and write for shortness $V_0 = a$, $V_1 = b$, $V_2 = c$, &c. Then

$$\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} = 2\,(ac - b^2),$$

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix} = 2\,(ae - 4bd + 3c^2),$$

$$\begin{bmatrix} 0 & \overset{+}{0} \\ 1 & 1 \end{bmatrix} = (ac - b^2), \text{ \&c.}$$

The functions on the second side are evidently hyperdeterminants such as are discussed in my memoir on Linear Transformations, and there is no difficulty in forming directly from the intermutant or commutant on the first side of the equation the symbol of derivation (in the sense of the memoir on Linear Transformations) from which the hyperdeterminant is obtained. Thus

$$\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \text{ is } \overline{12}^2 \,.\, UU, \qquad \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix} \text{ is } \overline{12}^4 \,.\, UU,$$

$$\begin{bmatrix} 0 & \overset{\dagger}{0} \\ 1 & 1 \end{bmatrix} \text{ is } \overline{12}\, U^{,0} U^{,1}, \qquad \begin{bmatrix} 0 & 0 & 0 & \overset{\dagger}{0} \\ 1 & 1 & 1 & 1 \end{bmatrix} \text{ is } \overline{12}^3 U^{,0} U^{,1},$$

each permutable column $\begin{matrix}0\\1\end{matrix}$ corresponding to a $\overline{12}$ [1] and a non-permutable column $\begin{matrix}\overset{\dagger}{0}\\1\end{matrix}$ changing UU into $U^{,0}U^{,1}$. Similarly

$$\begin{pmatrix}0 & 0\\1 & 1\\2 & 2\end{pmatrix} \text{ becomes } (\overline{12}\,.\,\overline{13}\,.\,\overline{23})^2\,.\,UUU,$$

$$\begin{pmatrix}0 & \overset{+}{0}\\1 & 1\\2 & 2\end{pmatrix} \text{ becomes } \overline{12}\,.\,\overline{13}\,.\,\overline{23}\;U^{,0}U^{,1}U^{,2}$$

$$\begin{pmatrix}0 & 0\\1 & 1\\2 & 2\\3 & 3\end{pmatrix} \text{ becomes } (\overline{12}\,.\,\overline{13}\,.\,\overline{14}\,.\,\overline{23}\,.\,\overline{24}\,.\,\overline{34})^2\;UUUU, \text{ \&c.}$$

The analogy would be closer if in the memoir on Linear Transformations, just as $\overline{12}$ is used to signify $\begin{vmatrix}\xi_1, & \eta_1\\ \xi_2, & \eta_2\end{vmatrix}$, $\overline{123}$ had been used to signify $\begin{vmatrix}\xi_1^2, & \xi_1\eta_1, & \eta_1^2\\ \xi_2^2, & \xi_2\eta_2, & \eta_2^2\\ \xi_3^2, & \xi_3\eta_3, & \eta_3^2\end{vmatrix}$ &c., for then $\begin{pmatrix}0 & 0\\1 & 1\\2 & 2\end{pmatrix}$ would have corresponded to $\overline{123}^2\,.\,UUU$, $\begin{pmatrix}0 & \overset{\dagger}{0}\\1 & 1\\2 & 2\end{pmatrix}$ to $\overline{123}\;U^{,0}U^{,1}U^{,2}$; and this would not only have been an addition of some importance to the theory, but would in some instances have facilitated the calculation of hyperdeterminants. The preceding remarks show that the intermutant

$$\begin{pmatrix}0 & 0 & 0\\ \underline{1} & \underline{1} & \overline{1}\\ 0 & 0 & 0\\ 1 & 1 & \overline{1}\end{pmatrix}$$

(where the first and fourth blanks in the last column are to be considered as belonging to the same set) is in the hyperdeterminant notation $(12\,.\,34)^2\,.\,(14\,.\,23)\,UUUU$.

[1] Viz. $\begin{matrix}0\\1\end{matrix}$ corresponds to $\overline{12}$ because 0 and 1 are the characters occupying the *first* and *second* blanks of a column. If 0 and 1 had been the characters occupying the *second* and *third* blanks in a column, the symbol would have been $\overline{23}$ and so on. It will be remembered, that the symbolic numbers 1, 2...... in the hyperdeterminant notation are merely introduced to distinguish from each other functions which are made identical after certain differentiations are performed.

It will, I think, illustrate the general theory to perform the development of the last-mentioned intermutant. We have

$$\begin{pmatrix} 0 & 0 & \underline{0} \\ \underline{1} & \underline{1} & 1 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & \overset{\dagger}{0} \\ \underline{1} & \underline{1} & 1 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 0 & \overset{\dagger}{0} \\ \underline{1} & \underline{1} & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 0 & \overset{\dagger}{1} \\ \underline{1} & \underline{1} & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & \overset{\dagger}{1} \\ \underline{1} & \underline{1} & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

$$= 2\left\{\begin{bmatrix} 0 & 0 & \overset{\dagger}{0} \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 0 & 0 & \overset{\dagger}{0} \\ 1 & 1 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 & \overset{\dagger}{0} \\ 1 & 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & \overset{\dagger}{1} \\ 1 & 1 & 1 \end{bmatrix}\right\}$$

$$= 2\{(ad - bc)^2 - 4(ac - b^2)(bd - c^2)\},$$

$$= 2(a^2d^2 + 4ac^3 + 4b^3d - 3b^2c^2 - 6abcd),$$

the different steps of which may be easily verified.

The following important theorem (which is, I believe, the same as a theorem of Mr Sylvester's, published in the *Philosophical Magazine*) is perhaps best exhibited by means of a simple example. Consider the intermutant

$$\begin{pmatrix} x & 1 \\ \underline{y} & 4 \\ x & 3 \\ y & 2 \end{pmatrix}$$

where in the first column the sets are distinguished as before by the horizontal bar, but in the second column the 1, 2 are to be considered as forming a set, and the 3, 4 as forming a second set. Then, partially expanding, the intermutant is

$$\begin{pmatrix} \overset{\dagger}{x} & 1 \\ y & 4 \\ x & 3 \\ y & 2 \end{pmatrix} - \begin{pmatrix} \overset{\dagger}{y} & 1 \\ x & 4 \\ x & 3 \\ y & 2 \end{pmatrix} - \begin{pmatrix} \overset{\dagger}{x} & 1 \\ y & 4 \\ y & 3 \\ x & 2 \end{pmatrix} + \begin{pmatrix} \overset{\dagger}{y} & 1 \\ x & 4 \\ y & 3 \\ x & 2 \end{pmatrix},$$

or, since entire horizontal lines may obviously be permuted,

$$\begin{pmatrix} \overset{\dagger}{x} & 1 \\ y & 2 \\ x & 3 \\ y & 4 \end{pmatrix} - \begin{pmatrix} \overset{\dagger}{y} & 1 \\ y & 2 \\ x & 3 \\ x & 4 \end{pmatrix} - \begin{pmatrix} \overset{\dagger}{x} & 1 \\ x & 2 \\ y & 3 \\ y & 4 \end{pmatrix} + \begin{pmatrix} \overset{\dagger}{y} & 1 \\ x & 2 \\ y & 3 \\ x & 4 \end{pmatrix};$$

and, observing that the 1, 2 form a permutable system as do also the 3, 4, the second and third terms vanish, while the first and fourth terms are equivalent to each other; we may therefore write

$$2\begin{pmatrix} \overset{\dagger}{x} & 1 \\ y & 2 \\ \overline{x} & \overline{3} \\ y & 4 \end{pmatrix} = \begin{pmatrix} x & 1 \\ y & 4 \\ \overline{x} & \overline{3} \\ y & 2 \end{pmatrix}$$

where on the first side of the equation the bar has been introduced into the second column, in order to show that *throughout* the equation the 1, 2 and the 3, 4 are to be considered as forming distinct sets.

Consider in like manner the expression

$$\begin{pmatrix} x & 1 \\ y & 7 \\ z & 6 \\ \overline{x} & 8 \\ y & 2 \\ z & 9 \\ \overline{x} & 4 \\ y & 5 \\ z & 3 \end{pmatrix}$$

where in the first column the sets are distinguished by the horizontal bars and in the second column the characters 1, 2, 3 and 4, 5, 6 and 7, 8, 9 are to be considered as belonging to distinct sets. The same reasoning as in the former case will show that this is a multiple of

$$\begin{pmatrix} x & 1 \\ y & 2 \\ z & 3 \\ \overline{x} & \overline{4} \\ y & 5 \\ z & 6 \\ \overline{x} & \overline{7} \\ y & 8 \\ z & 9 \end{pmatrix}$$

and to find the numerical multiplier it is only necessary to inquire in how many ways, in the expression first written down, the characters of the first column can be permuted so that x, y, z may go with 1, 2, 3 and with 4, 5, 6 and with 7, 8, 9. The order of the x, y, z in the second triad may be considered as arbitrary; but once assumed, it determines the place of one of the letters in the first triad; for instance, $x8$ and $z9$ determine $y7$. The first triad must therefore contain $x1$ and $z6$ or $x6$ and $z1$. Suppose the former, then the third triad must contain $z3$, but the remaining two combinations may be either $x4$, $y5$, or $x5$, $y4$. Similarly, if the first triad contained $x6$, $z1$, there would be two forms of the third triad, or a given form of the second triad gives four different forms. There are therefore in all 24 forms, or

$$24\left(\begin{array}{cc} x & 1 \\ y & 2 \\ z & 3 \\ \hline x & 4 \\ y & 5 \\ z & 6 \\ \hline x & 7 \\ y & 8 \\ z & 9 \end{array}\right)^{\dagger} = \left(\begin{array}{cc} x & 1 \\ y & 7 \\ z & 6 \\ \hline x & 8 \\ y & 2 \\ z & 9 \\ \hline x & 4 \\ y & 5 \\ z & 3 \end{array}\right)$$

where the bars in the second column on the first side show that *throughout* the equation 1, 2, 3 and 4, 5, 6 and 7, 8, 9 are to be considered as forming distinct sets. The above proof is in reality perfectly general, and it seems hardly necessary to render it so in terms.

To perceive the significance of the above equation it should be noticed that the first side is a product of determinants, viz.

$$24\begin{pmatrix} x & 1 \\ y & 2 \\ z & 3 \end{pmatrix}^{\dagger}\begin{pmatrix} x & 5 \\ y & 6 \\ z & 7 \end{pmatrix}^{\dagger}\begin{pmatrix} x & 7 \\ y & 8 \\ z & 9 \end{pmatrix}^{\dagger};$$

and if the second side be partially expanded by permuting the characters of the second column, each of the terms so obtained is in like manner a product of determinants, so that

$$24\begin{pmatrix} x & 1 \\ y & 2 \\ z & 3 \end{pmatrix}^{\dagger}\begin{pmatrix} x & 4 \\ y & 5 \\ z & 6 \end{pmatrix}^{\dagger}\begin{pmatrix} x & 7 \\ y & 8 \\ z & 9 \end{pmatrix}^{\dagger} = \begin{pmatrix} x & 1 \\ y & 7 \\ z & 6 \end{pmatrix}^{\dagger}\begin{pmatrix} x & 8 \\ y & 2 \\ z & 9 \end{pmatrix}^{\dagger}\begin{pmatrix} x & 4 \\ y & 5 \\ z & 3 \end{pmatrix}^{\dagger} \pm \&c.,$$

the permutations on the second side being the permutations *inter se* of 1, 2, 3, of 4, 5, 6, and of 7, 8, 9.

It is obvious that the preceding theorem is not confined to intermutants of two columns.

POSTSCRIPT.

I wish to explain as accurately as I am able, the extent of my obligations to Mr Sylvester in respect of the subject of the present memoir. The term permutant is due to him—intermutant and commutant are merely terms framed between us in analogy with permutant, and the names date from the present year (1851). The theory of commutants is given in my memoir in the *Cambridge Philosophical Transactions*, [12], and is presupposed in the memoir on Linear Transformations, [13, 14]. It will appear by the last-mentioned memoir that it was by representing the coefficients of a biquadratic function by $a = 1111$, $b = 1112 = 1121 = \&c.$, $c = 1122 = \&c.$, $d = 1222 = \&c.$, $e = 2222$, and forming the commutant $\begin{pmatrix} 1111 \\ 2222 \end{pmatrix}$ that I was led to the function $ae - 4bd + 3c^2$. The function $ace + 2bcd - ad^2 - b^2e - c^3$ or $\begin{vmatrix} a, & b, & c \\ b, & c, & d \\ c, & d, & e \end{vmatrix}$ is mentioned in the memoir on Linear Transformations, as brought into notice by Mr Boole. From the particular mode in which the coefficients a, b,... were represented by symbols such as 1111, &c., I did not perceive that the last-mentioned function could be expressed in the commutant notation. The notion of a permutant, in its most general sense, is explained by me in my memoir, "Sur les déterminants gauches," *Crelle*, t. XXXVII. pp. 93—96, [69]; see the paragraph (p. 94) commencing "On obtient ces fonctions, &c." and which should run as follows: "On obtient ces fonctions (dont je reprends ici la théorie) par les propriétés génerales d'un determinant défini comme suit. En exprimant &c.;" the sentence as printed being "......défini. Car en exprimant &c.," which confuses the sense. [The paragraph is printed correctly 69, p. 411.] Some time in the present year (1851) Mr Sylvester, in conversation, made to me the very important remark, that as one of a class the above-mentioned function,

$$ace + 2bcd - ad^2 - b^2e - c^3,$$

could be expressed in the commutant notation $\begin{pmatrix} 0 & 0 \\ 1 & 1 \\ 2 & 2 \end{pmatrix}$, viz. by considering $00 = a$, $01 = 10 = b$, $02 = 11 = 20 = c$, $12 = 21 = d$, $22 = e$; and the subject being thereby recalled to my notice, I found shortly afterwards the expression for the function

$$a^2d^2 + 4ac^3 + 4b^3d - 3b^2c^2 - 6abcd$$

(which cannot be expressed as a commutant) in the form of an intermutant, and I was thence led to see the identity, so to say, of the theory of hyperdeterminants, as given in the memoir on Linear Transformations, with the present theory of intermutants. It is understood between Mr Sylvester and myself, that the publication of the present memoir is not to affect Mr Sylvester's right to claim the origination, and to be considered as the first publisher of such part as may belong to him of the theory here sketched out.