# 109.

## ON THE RATIONALISATION OF CERTAIN ALGEBRAICAL EQUATIONS.

[From the *Cambridge and Dublin Mathematical Journal*, vol. VIII. (1853), pp. 97—101.]

SUPPOSE

$$x + y = 0, \quad x^2 = a, \quad y^2 = b;$$

then if we multiply the first equation by 1, $xy$, and reduce by the two others, we have

$$x + y = 0,$$
$$bx + ay = 0,$$

from which, eliminating $x$, $y$,

$$\begin{vmatrix} 1, & 1 \\ b, & a \end{vmatrix} = 0;$$

which is the equation between $a$ and $b$; or, considering $x$, $y$ as quadratic radicals, the rational equation between $x$, $y$. So if the original equation be multiplied by $x$, $y$, we have

$$a + xy = 0,$$
$$b + xy = 0;$$

or, eliminating 1, $xy$,

$$\begin{vmatrix} a, & 1 \\ b, & 1 \end{vmatrix} = 0,$$

which may be in like manner considered as the rational equation between $x$, $y$.

The preceding results are of course self-evident, but by applying the same process to the equations

$$x + y + z = 0, \quad x^2 = a, \quad y^2 = b, \quad z^2 = c,$$

we have results of some elegance. Multiply the equation first by 1, $yz$, $zx$, $xy$, reduce and eliminate the quantities $x$, $y$, $z$, $xyz$, we have the rational equation

$$\begin{vmatrix} . & 1 & 1 & 1 \\ 1 & . & c & b \\ 1 & c & . & a \\ 1 & b & a & . \end{vmatrix} = 0;$$

and again, multiply the equation by $x$, $y$, $z$, $xyz$, reduce and eliminate the quantities 1, $yz$, $zx$, $xy$, the result is

$$\begin{vmatrix} . & a & b & c \\ a & . & 1 & 1 \\ b & 1 & . & 1 \\ c & 1 & 1 & . \end{vmatrix} = 0,$$

which is of course equivalent to the preceding one (the two determinants are in fact identical in value), but the form is essentially different. The former of the two forms is that given in my paper "On a theorem in the Geometry of Position" (*Journal*, vol. II. [1841] p. 270 [1]): it was only very recently that I perceived that a similar process led to the latter of the two forms.

Similarly, if we have the equations

$$x + y + z + w = 0, \quad x^2 = a, \quad y^2 = b, \quad z^2 = c, \quad w^2 = d,$$

then multiplying by 1, $yz$, $zx$, $xy$, $xw$, $yw$, $zw$, $xyzw$, reducing and eliminating the quantities in the outside row,

we have the result

$$\begin{array}{cccccccc} x, & y, & z, & w, & yzw, & zwx, & wxy, & xyz \end{array}$$
$$\left|\begin{array}{ccc|c|ccc|c} 1 & 1 & 1 & 1 & . & . & . & . \\ \hline . & c & b & . & 1 & . & . & 1 \\ c & . & a & . & . & 1 & . & 1 \\ b & a & . & . & . & . & 1 & 1 \\ \hline d & . & . & a & . & 1 & 1 & . \\ . & d & . & b & 1 & . & 1 & . \\ . & . & d & c & 1 & 1 & . & . \\ \hline . & . & . & . & a & b & c & d \end{array}\right| = 0;$$

so if we multiply the equations by $x$, $y$, $z$, $w$, $yzw$, $zwx$, $wxy$, and $xyz$, reduce and eliminate the quantities in the outside row, we have the result

$$\begin{array}{c}
1,\ \ yz,\ \ zx,\ \ xy,\ \ xw,\ \ yw,\ \ zw,\ \ xyzw \\
\left|\begin{array}{c|ccc|ccc|c}
a & . & 1 & 1 & 1 & . & . & . \\
b & 1 & . & 1 & . & 1 & . & . \\
c & 1 & 1 & . & . & . & 1 & . \\
\hline
d & . & . & . & 1 & 1 & 1 & . \\
\hline
. & d & . & . & . & c & b & 1 \\
. & . & d & . & c & . & a & 1 \\
. & . & . & d & b & a & . & 1 \\
\hline
. & a & b & c & . & . & . & 1
\end{array}\right|
\end{array} = 0,$$

which however is not essentially distinct from the form before obtained, but may be derived from it by an interchange of lines and columns.

And in general for any *even* number of quadratic radicals the two forms are not essentially distinct, but may be derived from each other by interchanging lines and columns, while for an *odd* number of quadratic radicals the two forms cannot be so derived from each other, but are essentially distinct.

I was indebted to Mr Sylvester for the remark that the above process applies to radicals of a higher order than the second. To take the simplest case, suppose

$$x + y = 0, \quad x^3 = a, \quad y^3 = b;$$

and multiply first by 1, $x^2y$, $xy^2$; this gives

$$\begin{array}{rcl}
x + y & . & = 0 \\
. & ay + x^2y^2 & = 0 \\
bx & . \ + x^2y^2 & = 0\,;
\end{array}$$

or, eliminating,

$$\left|\begin{array}{ccc} 1 & 1 & . \\ . & a & 1 \\ b & . & 1 \end{array}\right| = 0\,;$$

next multiply by $x$, $y$, $x^2y^2$; this gives

$$\begin{array}{rcl}
x^2 & . \ + xy & = 0 \\
. & y^2 + xy & = 0 \\
bx^2 + ay^2 & . & = 0\,;
\end{array}$$

or, eliminating,

$$\left|\begin{array}{ccc} 1 & . & 1 \\ . & 1 & 1 \\ b & a & . \end{array}\right| = 0\,;$$

and lastly, multiply by $x^2$, $y^2$, $xy$; this gives

$$\begin{array}{rcl}
a + x^2y & . & = 0 \\
b & . \ + xy^2 & = 0 \\
. & x^2y + xy^2 & = 0\,;
\end{array}$$

or, eliminating,

$$\left|\begin{array}{ccc} a & 1 & . \\ b & . & 1 \\ . & 1 & 1 \end{array}\right| = 0;$$

where it is to be remarked that the second and third forms are not essentially distinct, since the one may be derived from the other by the interchange of lines and columns.

Applying the preceding process to the system

$$x + y + z = 0, \quad x^3 = a, \quad y^3 = b, \quad z^3 = c;$$

multiply first by 1, $xyz$, $x^2y^2z^2$, $x^2z$, $y^2x$, $z^2y$, $x^2y$, $y^2z$, $z^2x$, reduce and eliminate the quantities in the outside row,

the result is

$$\begin{array}{c} \begin{array}{ccccccccc} x, & y, & z, & y^2z^2, & x^2yz, & y^2zx, & z^2xy, & z^2x^2, & x^2y^2 \end{array} \\ \left|\begin{array}{ccc|ccc|ccc} 1 & 1 & 1 & . & . & . & . & . & . \\ \hline . & . & . & 1 & 1 & 1 & . & . & . \\ \hline . & . & . & . & . & . & a & b & c \\ \hline . & . & a & 1 & . & . & . & 1 & . \\ b & . & . & . & 1 & . & . & . & 1 \\ . & c & . & . & . & 1 & 1 & . & . \\ \hline . & a & . & 1 & . & . & . & . & 1 \\ . & . & b & . & 1 & . & 1 & . & . \\ c & . & . & . & . & 1 & . & 1 & . \end{array}\right| \end{array} = 0;$$

next multiply by $x$, $y$, $z$, $y^2z^2$, $z^2x^2$, $x^2y^2$, $x^2yz$, $y^2zx$, $z^2xy$, reduce and eliminate the quantities in the outside row,

the result is

$$\begin{array}{c} \begin{array}{ccccccccc} x^2, & y^2, & z^2, & yz, & zx, & xy, & xy^2z^2, & yz^2x^2, & zx^2y^2 \end{array} \\ \left|\begin{array}{ccc|ccc|ccc} 1 & . & . & . & 1 & 1 & . & . & . \\ . & 1 & . & 1 & . & 1 & . & . & . \\ . & . & 1 & 1 & 1 & . & . & . & . \\ \hline . & c & b & . & . & . & 1 & . & . \\ c & . & a & . & . & . & . & 1 & . \\ b & a & . & . & . & . & . & . & 1 \\ \hline . & . & . & a & . & . & . & 1 & 1 \\ . & . & . & . & b & . & 1 & . & 1 \\ . & . & . & . & . & c & 1 & 1 & . \end{array}\right| \end{array} = 0;$$

lastly, multiply by $x^2$, $y^2$, $z^2$, $yz$, $zx$, $xy$, $xy^2z^2$, $yz^2x^2$, $xy^2z^2$, reduce and eliminate the quantities in the outside row,

the result is

$$\begin{array}{ccccccccc}
1 & xyz, & x^2y^2z^2, & yz^2, & zx^2, & xy^2, & y^2z, & z^2x, & x^2y
\end{array}$$

$$\left|\begin{array}{c|c|c|ccc|ccc}
a & . & . & . & 1 & . & . & . & 1 \\
b & . & . & . & . & 1 & 1 & . & . \\
c & . & . & 1 & . & . & . & 1 & . \\
\hline
. & 1 & . & 1 & . & . & 1 & . & . \\
. & 1 & . & . & 1 & . & . & 1 & . \\
. & 1 & . & . & . & 1 & . & . & 1 \\
\hline
. & . & 1 & . & . & c & . & b & . \\
. & . & 1 & a & . & . & . & . & c \\
. & . & 1 & . & b & . & a & . & .
\end{array}\right| = 0;$$

where, as in the case of two cubic radicals, two forms, viz. the first and third forms of the rational equation, are not essentially distinct, but may be derived from each other by interchanging lines and columns.

And in general, whatever be the number of cubic radicals, two of the three forms are not essentially distinct, but may be derived from each other by interchanging lines and columns.