116.

CORRECTION OF TWO THEOREMS RELATING TO THE PORISM OF THE IN-AND-CIRCUMSCRIBED POLYGON.

[From the *Philosophical Magazine*, vol. VI. (1853), pp. 376—377.]

THE two theorems in my "Note on the Porism of the in-and-circumscribed Polygon" (see August Number), [115], are erroneous, the mistake arising from my having inadvertently assumed a wrong formulæ for the addition of elliptic integrals. The first of the two theorems (which, in fact, includes the other as a particular case) should be as follows:—

THEOREM. The condition that there may be inscribed in the conic $U=0$ an infinity of n-gons circumscribed about the conic $V=0$, depends upon the development in ascending powers of ξ of the square root of the discriminant of $\xi U+V$; viz. if this square root be

$$A+B\xi+C\xi^2+D\xi^3+E\xi^4+F\xi^5+G\xi^6+H\xi^7+\dots,$$

then for $n=3$, 5, 7, &c. respectively, the conditions are

$$|C|=0,\quad \begin{vmatrix} C, & D \\ D, & E \end{vmatrix}=0,\quad \begin{vmatrix} C, & D, & E \\ D, & E, & F \\ E, & F, & G \end{vmatrix}=0, \text{ \&c.};$$

and for $n=4$, 6, 8, &c. respectively, the conditions are

$$|D|=0,\quad \begin{vmatrix} D, & E \\ E, & F \end{vmatrix}=0,\quad \begin{vmatrix} D, & E, & F \\ D, & F, & G \\ F, & G, & H \end{vmatrix}=0, \text{ \&c.}$$

The examples require no correction; since for the triangle and the quadrilateral respectively, the conditions are (as in the erroneous theorem) $C=0$, $D=0$.

The second theorem gives the condition in the case where the conics are replaced by the circles $x^2+y^2-R^2=0$ and $(x-a)^2+y^2-r^2=0$, the discriminant being in this case

$$-(1+\xi)\{r^2+\xi(r^2+R^2-a^2)+\xi^2R^2\}.$$

As a very simple example, suppose that the circles are concentric, or assume $a=0$; the square root of the discriminant is here

$$(1+\xi)\sqrt{r^2+R^2\xi};$$

and putting for shortness $\frac{R^2}{r^2}=\alpha$, we may write

$$A+B\xi+\ldots=(1+\xi)\sqrt{1+\alpha\xi},$$

that is, $A=1$, $B=\frac{1}{2}\alpha+1$, $C=-\frac{1}{8}\alpha^2+\frac{1}{2}\alpha^2$, $D=\frac{1}{16}\alpha^3-\frac{1}{8}\alpha^2$, $E=-\frac{5}{128}\alpha^4+\frac{1}{16}\alpha^3$, &c.; thus in the case of the pentagon,

$$CE-D^2=\tfrac{1}{1024}\alpha^4\{(\alpha-4)(5\alpha-8)-4(\alpha-2)^2\}$$

$$=\tfrac{1}{1024}\alpha^4(\alpha^2-12\alpha+16),$$

and the required condition therefore is

$$\alpha^2-12\alpha+16=$$

It is clear that, in the case in question,

$$\frac{r}{R}=\cos 36^\circ=\tfrac{1}{4}(\sqrt{5}+1),$$

that is, $$\frac{R}{r}=\sqrt{5}-1, \text{ or } (R+r)^2-5r^2=0,$$

viz. $$(\sqrt{\alpha}+1)^2-5=0, \text{ or } \alpha+2\sqrt{\alpha}-4=0,$$

the rational form of which is

$$\alpha^2-12\alpha+16=0,$$

and we have thus a verification of the theorem for this particular case.

2 *Stone Buildings, Oct.* 10, 1853.