117.

NOTE ON THE INTEGRAL $\int dx \div \sqrt{(m-x)(x+a)(x+b)(x+c)}$.

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IF in the formulæ of my "Note on the Porism of the in-and-circumscribed Polygon," [115], it is assumed that

$$U = x^2 + y^2 + z^2 + \frac{1}{m}(ax^2 + by^2 + cz^2)$$

$$V = \qquad ax^2 + by^2 + cz^2,$$

and if a new parameter ω connected with the parameter w by the equation

$$w = \frac{\omega m}{m - \omega}$$

be made use of instead of w, then

$$wU + V = \frac{m}{m-\omega}\{\omega(x^2+y^2+z^2) + ax^2 + by^2 + cz^2\};$$

and thus the equation $wU + V = 0$, viz. the equation

$$\omega(x^2+y^2+z^2) + ax^2 + by^2 + cz^2 = 0,$$

is precisely of the same form as that considered in my "Note on the Geometrical Representation of the Integral $\int dx \div \sqrt{(x+a)(x+b)(x+c)}$," [113.] Moreover, introducing instead of ξ a quantity η, such that

$$\xi = \frac{m\eta}{m-\eta},$$

then

$$\frac{d\xi}{\sqrt{\square\xi}} = \frac{\sqrt{m}\, d\eta}{\sqrt{(m-\eta)(a+\eta)(b+\eta)(c+\eta)}}.$$

Also $\xi=\infty$ gives $\eta=m$, the integral to be considered is therefore

$$\Pi,\eta=\int_m \frac{\sqrt{m}\,d\eta}{\sqrt{(m-\eta)(a+\eta)(b+\eta)(c+\eta)}};$$

i.e. if in the paper last referred to the parameter ∞ had been throughout replaced by the parameter m, the integral

$$\Pi\eta=\int_\infty \frac{d\eta}{\sqrt{(a+\eta)(b+\eta)(c+\eta)}}$$

would have had to be replaced by the integral Π,η. It is, I think, worth while to reproduce for this more general case a portion of the investigations of the paper in question, for the sake of exhibiting the rational and integral form of the algebraical equation corresponding to the transcendental equation $\pm\Pi,k\pm\Pi,p\pm\Pi,\theta=0$. Consider the point $\xi,\ \eta,\ \zeta$ on the conic $m(x^2+y^2+z^2)+ax^2+by^2+cz^2=0$, the equation of the tangent at this point is

$$(m+a)\,\xi x+(m+b)\,\eta y+(m+c)\,\zeta z=0;$$

and if θ be the other parameter of this line, then the line touches

$$\theta(x^2+y^2+z^2)+ax^2+by^2+cz^2=0;$$

or we have

$$\frac{(m+a)^2\,\xi^2}{\theta+a}+\frac{(m+b)^2\,\eta^2}{\theta+b}+\frac{(m+c)^2\,\zeta^2}{\theta+c}=0;$$

and combining this with

$$(m+a)\,\xi^2+(m+b)\,\eta^2+(m+c)\,\zeta^2=0,$$

we have

$$\begin{aligned}\xi:\eta:\zeta=&\ \sqrt{b-c}\ \sqrt{a+\theta}\,\sqrt{b+m}\,\sqrt{c+m}\\ &:\sqrt{(c-a)}\,\sqrt{b+\theta}\,\sqrt{c+m}\,\sqrt{a+m}\\ &:\sqrt{(a-b)}\,\sqrt{c+\theta}\,\sqrt{a+m}\,\sqrt{b+m}\end{aligned}$$

for the coordinates of the point P. Substituting these for $x,\ y,\ z$ in the equation of the line PP' (the parameters of which are $p,\ k$), viz. in

$$x\sqrt{b-c}\,\sqrt{(a+k)(a+p)}+y\sqrt{c-a}\,\sqrt{(b+k)(b+p)}+z\sqrt{a-b}\,\sqrt{c+k}\,\sqrt{c+p}=0,$$

we have

$$(b-c)\frac{\sqrt{(a+p)(a+k)(a+\theta)}}{\sqrt{a+m}}+(c-a)\frac{\sqrt{(b+p)(b+k)(b+\theta)}}{\sqrt{b+m}}$$
$$+(a-b)\frac{\sqrt{(c+p)(c+k)(c+\theta)}}{\sqrt{c+m}}=0,$$

which is to be replaced by
$$\frac{(a+p)(a+k)(a+\theta)}{a+m} = (\lambda+\mu a)^2,$$
$$\frac{(b+p)(b+k)(b+\theta)}{b+m} = (\lambda+\mu b)^2,$$
$$\frac{(c+p)(c+k)(c+\theta)}{c+m} = (\lambda+\mu c)^2.$$

These equations give, omitting the common factor $(a+m)(b+m)(c+m)$,

$$\begin{aligned}\lambda^2 = \; & m^2(abc+pk\theta)\\ & +m\{-abc(p+k+\theta)+pk\theta(a+b+c)\}\\ & + \{abc(k\theta+\theta p+kp)+pk\theta(bc+ca+ab)\},\end{aligned}$$

$$\begin{aligned}2\lambda\mu = \; & m^2\{-(bc+ca+ab)+(k\theta+\theta p+pk)\}\\ & +m\{-abc-pk\theta+(bc+ca+ab)(p+k+\theta)+(k\theta+\theta p+pk)(a+b+c)\}\\ & + \{abc(p+k+\theta)-pk\theta(a+b+c)\},\end{aligned}$$

$$\begin{aligned}\mu^2 = \; & m^2\{a+b+c+p+k+\theta\}\\ & +m\{(bc+ca+ab)-(k\theta+\theta p+pk)\}\\ & + \quad abc+pk\theta\,;\end{aligned}$$

and substituting in $4\lambda^2 . \mu^2 - (2\lambda\mu)^2 = 0$, we have the relation required. To verify that the equation so obtained is in fact the algebraical equivalent of the transcendental equation, it is only necessary to remark, that the values of λ^2, μ^2 are unaltered, and that of $\lambda\mu$ only changes its sign when a, b, c, m and p, k, θ, $-m$ are interchanged; and so this change will not affect the equation obtained by substituting in the equation $4\lambda^2 . \mu^2 - (2\lambda\mu)^2 = 0$. Hence precisely the same equation would be obtained by eliminating L, M from
$$(k+a)(k+b)(k+c) = (L+Mk)^2(m-k),$$
$$(p+a)(p+b)(p+c) = (L+Mp)^2(m-p),$$
$$(\theta+a)(\theta+b)(\theta+c) = (L+M\theta)^2(\theta-p)\,;$$

or, putting $(L+Mk)(m-k) = \alpha+\beta k+\gamma k^2$, by eliminating α, β, γ from
$$(m-k)(k+a)(k+b)(k+c) = (\alpha+\beta k+\gamma k^2)^2,$$
$$(m-p)(p+a)(p+b)(p+c) = (\alpha+\beta p+\gamma p^2)^2,$$
$$(m-\theta)(\theta+a)(\theta+b)(\theta+c) = (\alpha+\beta\theta+\gamma\theta^2)^2,$$
$$0 = (\alpha+\beta m+\gamma m^2)^2,$$

which by Abel's theorem show that p, k, θ are connected by the transcendental equation above mentioned.

2 *Stone Buildings, July* 9, 1853.