# 139.

# AN INTRODUCTORY MEMOIR UPON QUANTICS.

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1. THE term Quantics is used to denote the entire subject of rational and integral functions, and of the equations and loci to which these give rise; the word "quantic" is an adjective, meaning *of such a degree*, but may be used substantively, the noun understood being (unless the contrary appear by the context) function; so used the word admits of the plural "quantics."

The quantities or symbols to which the expression "degree" refers, or (what is the same thing) in regard to which a function is considered as a quantic, will be spoken of as "facients." A quantic may always be considered as being, in regard to its facients, homogeneous, since to render it so, it is only necessary to introduce as a facient unity, or some symbol which is to be ultimately replaced by unity; and in the cases in which the facients are considered as forming two or more distinct sets, the quantic may, in like manner, be considered as homogeneous in regard to each set separately.

2. The expression "an equation," used without explanation, is to be understood as meaning the equation obtained by putting any quantic equal to zero. I make no absolute distinction between the words "degree" and "order" as applied to an equation or system of equations, but I shall in general speak of the order rather than the degree. The equations of a system may be independent, or there may exist relations of connexion between the different equations of the system; the subject of a system of equations so connected together is one of extreme complexity and difficulty. It will be sufficient to notice here, that in any system whatever of equations, assuming only that the equations are not more than sufficient to determine the ratios of the facients, and joining to the system so many linear equations between the facients as will render the ratios of the facients determinate, the order of the system is the same thing as the order of the equation which determines any one of these ratios; it is clear that for a single equation the order so determined is nothing else than the order of the equation.

3. An equation or system of equations represents, or is represented by a locus. This assumes that the facients depend upon quantities $x, y, \ldots$ the coordinates of a point in space; the entire series of points, the coordinates of which satisfy the equation or system of equations, constitutes the locus. To avoid complexity, it is proper to take the facients themselves as coordinates, or at all events to consider these facients as linear functions of the coordinates; this being the case, the order of the locus will be the order of the equation, or system of equations.

4. I have spoken of the *coordinates* of a *point* in *space*. I consider that there is an ideal space of any number of dimensions, but of course, in the ordinary acceptation of the word, space is of three dimensions; however, the plane (the space of ordinary plane geometry) is a space of two dimensions, and we may consider the line as a space of one dimension. I do not, it should be observed, say that the only idea which can be formed of a space of two dimensions is the plane, or the only idea which can be formed of space of one dimension is the line; this is not the case. To avoid complexity, I will take the case of plane geometry rather than geometry of three dimensions; it will be unnecessary to speak of space, or of the number of its dimensions, or of the plane, since we are only concerned with space of two dimensions, viz. the plane; I say, therefore, simply that $x, y, z$ are the coordinates of a point (strictly speaking, it is the ratios of these quantities which are the coordinates, and the quantities $x, y, z$ themselves are indeterminates, i.e. they are only determinate to a common factor *près*, so that in assuming that the coordinates of a point are $\alpha, \beta, \gamma$, we mean only that $x:y:z=\alpha:\beta:\gamma$, and we never as a result obtain $x, y, z=\alpha, \beta, \gamma$, but only $x:y:z=\alpha:\beta:\gamma$; but this being once understood, there is no objection to speaking of $x, y, z$ as coordinates). Now the notions of coordinates and of a point are merely relative; we may, if we please, consider $x:y:z$ as the parameters of a curve containing two variable parameters; such curve becomes of course determinate when we assume $x:y:z=\alpha:\beta:\gamma$, and this very curve is nothing else than the point whose coordinates are $\alpha, \beta, \gamma$, or as we may for shortness call it, the point $(\alpha, \beta, \gamma)$. And if the coordinates $(x, y, z)$ are connected by an equation, then giving to these coordinates the entire system of values which satisfy the equation, the locus of the points corresponding to these values is the locus representing or represented by the equation; this of course fixes the notion of a curve of any order, and in particular the notion of a line as the curve of the first order.

The theory includes, as a very particular case, the ordinary theory of reciprocity in plane geometry; we have only to say that the word "point" shall mean "line," and the word "line" shall mean "point," and that expressions properly or primarily applicable to a point and a line respectively shall be construed to apply to a line and a point respectively, and any theorem (assumed of course to be a purely descriptive one) relating to points and lines will become a corresponding theorem relating to lines and points; and similarly with regard to curves of a higher order, when the ideas of reciprocity applicable to these curves are properly developed.

5. A quantic of the degrees $m, m' \ldots$ in the sets $(x, y \ldots)$, $(x', y' \ldots)$ &c. will for the most part be represented by a notation such as

$$(*\between x, y \ldots \overset{m}{\between} x', y' \ldots \overset{m'}{)} \ldots),$$

where the mark * may be considered as indicative of the absolute generality of the quantic; any such quantic may of course be considered as the sum of a series of terms $x^\alpha y^\beta \ldots x'^{\alpha'} y'^{\beta'} \ldots$, &c. of the proper degrees in the different sets respectively, each term multiplied by a coefficient; these coefficients may be mere numerical multiples of single letters or elements such as $a, b, c, \ldots$, or else functions (in general rational and integral functions) of such elements; this explains the meaning of the expression "the elements of a quantic": in the case where the coefficients are mere numerical multiples of the elements, we may in general speak indifferently of the elements, or of the coefficients. I have said that the coefficients may be numerical multiples of single letters or elements such as $a, b, c, \ldots$; by the appropriate numerical coefficient of a term $x^\alpha y^\beta \ldots x'^{\alpha'} y'^{\beta'} \ldots$, I mean the coefficient of this term in the expansion of

$$(x+y\ldots)^m (x'+y'\ldots)^{m'}\ldots);$$

and I represent by the notation

$$(a,\ b,\ldots \between x,\ y,\ldots)^m \between x',\ y',\ldots)^{m'}\ldots),$$

a quantic in which each term is multiplied as well by its appropriate numerical coefficient as by the literal coefficient or element which belongs to it in the set $(a, b, \ldots)$ of literal coefficients or elements. On the other hand, I represent by the notation

$$(a,\ b,\ldots \between x,\ y,\ldots \overset{m}{\between} x',\ y',\ldots \overset{m'}{)}\ldots),$$

a quantic in which each term is multiplied only by the literal coefficient or element which belongs to it in the set $(a, b, \ldots)$ of literal coefficients or elements. And a like distinction applies to the case where the coefficients are functions of the elements $(a, b, \ldots)$.

6. I consider now the quantic

$$(* \between x,\ y,\ldots)^m \between x',\ y',\ldots)^{m'}\ldots),$$

and selecting any two facients of the same set, e.g. the facients $x$, $y$, I remark that there is always an operation upon the elements, tantamount as regards the quantic to the operation $x\partial_y$; viz. if we differentiate with respect to each element, multiply by proper functions of the elements and add, we obtain the same result as by differentiating with $\partial_y$ and multiplying by $x$. The simplest example will show this as well as a formal proof; for instance, as regards $3ax^2 + bxy + 5cy^2$ (the numerical coefficients are taken haphazard), we have $\frac{1}{3}b\partial_a + 10c\partial_b$ tantamount to $x\partial_y$; as regards $a(x-\alpha y)(x-\beta y)$, we have $-a(\alpha+\beta)\partial_a + \alpha^2\partial_\alpha + \beta^2\partial_\beta$ tantamount to $x\partial_y$, and so in any other case. I represent by $\{x\partial_y\}$ the operation upon the elements tantamount to $x\partial_y$, and I write down the series of operations

$$\{x\partial_y\} - x\partial_y, \ldots \quad \{x'\partial_{y'}\} - x'\partial_{y'}, \ldots$$

where $x$, $y$ are considered as being successively replaced by every permutation of two different facients of the set $(x, y, \ldots)$; $x'$, $y'$ as successively replaced by every permutation of two different facients of the set $(x', y', \ldots)$, and so on; this I call an entire system, and

I say that it is made up of partial systems corresponding to the different facient sets respectively; it is clear from the definition that the quantic is reduced to zero by each of the operations of the entire system. Now, besides the quantic itself, there are a variety of other functions which are reduced to zero by each of the operations of the entire system; any such function is said to be a covariant of the quantic, and in the particular case in which it contains only the elements, an invariant. (It would be allowable to define as a covariant *quoad any set or sets*, a function which is reduced to zero by each of the operations of the corresponding partial system or systems, but this is a point upon which it is not at present necessary to dwell.)

7. The definition of a covariant may however be generalized in two directions: we may instead of a single quantic consider two or more quantics; the operations $\{x\partial_y\}$, although represented by means of the same symbols $x$, $y$ have, as regards the different quantics, different meanings, and we may form the sum $\Sigma\{x\partial_y\}$, where the summation refers to the different quantics: we have only to consider in place of the system before spoken of, the system

$$\Sigma\{x\partial_y\} - x\partial_y, \ldots ; \ \Sigma\{x'\partial_{y'}\} - x'\partial_{y'}, \ldots \text{ \&c. \&c.},$$

and we obtain the definition of a covariant of two or more quantics.

Again, we may consider in connexion with each set of facients any number of new sets, the facients in any one of these new sets corresponding each to each with those of the original set; and we may admit these new sets into the covariant. This gives rise to a sum $S\{x\partial_y\}$, where the summation refers to the entire series of corresponding sets. We have in place of the system spoken of in the original definition, to consider the system

$$\{x\partial_y\} - S(x\partial_y), \ldots \{x'\partial_{y'}\} - S(x'\partial_{y'}), \ldots \text{ \&c. \&c.},$$

or if we are dealing with two or more quantics, then the system

$$\Sigma\{x\partial_y\} - S(x\partial_y), \ldots ; \ \Sigma\{x'\partial_{y'}\} - S(x'\partial_{y'}), \ldots \text{ \&c. \&c.},$$

and we obtain the generalized definition of a covariant.

8. A covariant has been defined simply as a function reduced to zero by each of the operations of the entire system. But in dealing with given quantics, we may without loss of generality consider the covariant as a function of the like form with the quantic, i.e. as being a rational and integral function homogeneous in regard to the different sets separately, and as being also a rational and integral function of the elements. In particular in the case where the coefficients are mere numerical multiples of the elements, the covariant is to be considered as a rational and integral function homogeneous in regard to the different sets separately, and also homogeneous in regard to the coefficients or elements. And the term "covariant" includes, as already remarked, "invariant."

It is proper to remark, that if the same quantic be represented by means of different sets of elements, then the symbols $\{x\partial_y\}$ which correspond to these different forms of the same quantic are mere transformations of each other, i.e. they become in virtue of the relations between the different sets of elements identical.

9. What precedes is a return to and generalization of the method employed in the first part of the memoir published in the *Camb. Math. Jour.*, t. IV. [1845], and *Camb. and Dubl. Math. Jour.*, t. I. [1846], under the title "On Linear Transformations," [13 and 14], and Crelle, t. XXX. [1846], under the title "Mémoire sur les Hyperdéterminants," [*16], and which I shall refer to as my original memoir. I there consider in fact the invariants of a quantic

$$(* \between x_1, x_2 \dots x_m \between y_1, y_2 \dots y_m) \dots),$$

linear in regard to $n$ sets each of them of $m$ facients, and I represent the coefficients of a term $x_r y_s z_t \dots$ by $rst \dots$; there is no difficulty in seeing that $\alpha$, $\beta$ being any two different numbers out of the series 1, 2, ... $m$, the operation $\{x_\beta \partial_{x_\alpha}\}$ is identical with the operation

$$\Sigma\Sigma \dots \left(\alpha st \dots \frac{d}{d\beta st \dots}\right),$$

where the summations refer to $s, t, \dots$ which pass respectively from 1 to $m$, both inclusive; and the condition that a function, assumed to be an invariant, i.e. to contain only the coefficients, may be reduced to zero by the operation $\{x_\beta \partial_{x_\alpha}\} - x_\beta \partial_{x_\alpha}$, is of course simply the condition that such function may be reduced to zero by the operation $\{x_\beta \partial_{x_\alpha}\}$; the condition in question is therefore the same thing as the equation

$$\Sigma\Sigma \dots \left(\alpha st \dots \frac{d}{d\beta st}\right) u = 0$$

of my original memoir.

10. But the definition in the present memoir includes also the method made use of in the second part of my original memoir. This method is substantially as follows: consider for simplicity a quantic $U =$

$$(* \between x, y, \dots)^m$$

containing only the single set $(x, y \dots)$, and let $U_1, U_2 \dots$ be what the quantic becomes when the set $(x, y \dots)$ is successively replaced by the sets $(x_1, y_1, \dots)$, $(x_2, y_2, \dots)$, ... the number of these new sets being equal to or greater than the number of facients in the set. Suppose that $A, B, C, \dots$ are any of the determinants

$$\left\| \begin{matrix} \partial_{x_1}, & \partial_{x_2}, & \partial_{x_3}, & \dots \\ \partial_{y_1}, & \partial_{y_2}, & \partial_{y_3}, & \\ \vdots & & & \end{matrix} \right\|,$$

then forming the derivative

$$A^p B^q C^r \dots U_1 U_2 \dots,$$

where $p, q, r \dots$ are any positive integers, the function so obtained is a covariant involving the sets $(x_1, y_1, \dots)$, $(x_2, y_2, \dots)$ &c.; and if after the differentiations we replace these sets by the original set $(x, y, \ldots)$, we have a covariant involving only the original set $(x, y, \ldots)$ and of course the coefficients of the quantic. It is in fact easy to show that any such derivative is a covariant according to the definition given in this Memoir. But to do this some preliminary explanations are necessary.

11. I consider any two operations $P$, $Q$, involving each or either of them differentiations in respect of variables contained in the other of them. It is required to investigate the effect of the operation $P \,.\, Q$, where the operation $Q$ is to be in the first place performed upon some operand $\Omega$, and the operation $P$ is then to be performed on the operand $Q\Omega$. Suppose that $P$ involves the differentiations $\partial_a, \partial_b, \ldots$ in respect of variables $a, b, \ldots$ contained in $Q$ and $\Omega$, we must as usual in the operation $P$ replace $\partial_a, \partial_b, \ldots$ by $\partial_a + \partial'_a, \partial_b + \partial'_b, \ldots$ where the unaccentuated symbols operate only upon $\Omega$, and the accentuated symbols operate only upon $Q$. Suppose that $P$ is expanded in ascending powers of the symbols $\partial'_a, \partial'_b, \ldots$, viz. in the form $P + P_1 + P_2 +$ &c., we have first to find the values of $P_1 Q$, $P_2 Q$, &c., by actually performing upon $Q$ as operand the differentiations $\partial'_a, \partial'_b \ldots$. The symbols $PQ$, $P_1 Q$, $P_2 Q$, &c. will then contain only the differentiations $\partial_a, \partial_b, \ldots$ which operate upon $\Omega$, and the meaning of the expression being once understood, we may write

$$P \,.\, Q = PQ + P_1 Q + P_2 Q + \text{\&c.}$$

In particular if $P$ be a linear function of $\partial_a, \partial_b, \ldots$, we have to replace $P$ by $P + P_1$, where $P_1$ is the same function of $\partial'_a, \partial'_b, \ldots$ that $P$ is of $\partial_a, \partial_b, \ldots$, and it is therefore clear that we have in this case

$$P \,.\, Q = PQ + P(Q),$$

where on the right-hand side in the term $PQ$ the differentiations $\partial_a, \partial_b, \ldots$ are considered as not in anywise affecting the symbol $Q$, while in the term $P(Q)$ these differentiations, or what is the same thing, the operation $P$, is considered to be performed upon $Q$ as operand.

Again, if $Q$ be a linear function of $a, b, c, \ldots$, then $P_2 Q = 0$, $P_3 Q = 0$, &c., and therefore $P \,.\, Q = PQ + P_1 Q$; and I shall in this case also (and consequently whenever $P_2 Q = 0$, $P_3 Q = 0$, &c.) write

$$P \,.\, Q = PQ + P(Q),$$

where on the right-hand side in the term $PQ$ the differentiations $\partial_a, \partial_b, \ldots$ are considered as not in anywise affecting the symbol $Q$, while the term $P(Q)$ is in each case what has been in the first instance represented by $P_1 Q$.

We have in like manner, if $Q$ be a linear function of $\partial_a, \partial_b, \partial_c, \ldots$, or if $P$ be a linear function of $a, b, c, \ldots$,

$$Q \,.\, P = QP + Q(P):$$

and from the two equations (since obviously $PQ = QP$) we derive

$$P \,.\, Q - Q \,.\, P = P(Q) - Q(P),$$

which is the form in which the equations are most frequently useful.

12. I return to the expression

$$A^p B^q C^r \dots U_1 U_2 \dots,$$

and I suppose that after the differentiations the sets $(x_1, y_1, \dots)$, $(x_2, y_2, \dots)$, &c. are replaced by the original set $(x, y, \dots)$. To show that the result is a covariant, we must prove that it is reduced to zero by an operation $\mathfrak{D} =$

$$\{x\partial_y\} - x\partial_y.$$

It is easy to see that the change of the sets $(x_1, y_1, \dots)$, $(x_2, y_2, \dots)$, &c. into the original set $(x, y, \dots)$ may be deferred until after the operation $\mathfrak{D}$, provided that $x\partial_y$ is replaced by $x_1\partial_{y_1} + x_2\partial_{y_2} + \dots$, or if we please by $Sx\partial_y$; we must therefore write $\mathfrak{D} = \{x\partial_y\} - Sx\partial_y$. Now in the equation

$$A \,.\, \mathfrak{D} - \mathfrak{D} \,.\, A = A\,(\mathfrak{D}) - \mathfrak{D}\,(A),$$

where, as before, $A\,(\mathfrak{D})$ denotes the result of the operation $A$ performed upon $\mathfrak{D}$ as operand, and similarly $\mathfrak{D}\,(A)$ the result of the operation $\mathfrak{D}$ performed upon $A$ as operand, we see first that $A\,(\mathfrak{D})$ is a determinant two of the lines of which are identical, it is therefore equal to zero; and next, since $\mathfrak{D}$ does not involve any differentiations affecting $A$, that $\mathfrak{D}\,(A)$ is also equal to zero. Hence $A \,.\, \mathfrak{D} - \mathfrak{D} \,.\, A = 0$ or $A$ and $\mathfrak{D}$ are convertible. But in like manner $\mathfrak{D}$ is convertible with $B$, $C$, &c., and consequently $\mathfrak{D}$ is convertible with $A^p B^q C^r \dots$. Now $\mathfrak{D}\, U_1 U_3 \dots = 0$; hence

$$\mathfrak{D} \,.\, A^p B^q C^r \dots U_1 U_2 \dots = 0,$$

or $A^p B^q C^r \dots U_1 U_2 \dots$ is a covariant, the proposition which was to be proved.

13. I pass to a theorem which leads to another method of finding the covariants of a quantic. For this purpose I consider the quantic

$$(*\,\check{\gamma}\, x, y \dots \overset{m}{\check{\gamma}}\, x', y' \dots \overset{m'}{)} \dots),$$

the coefficients of which are mere numerical multiples of the elements $(a, b, c, \dots)$; and in connexion with this quantic I consider the linear functions $\xi x + \eta y \dots$, $\xi' x' + \eta' y' \dots$, which treating $(\xi, \eta, \dots)$, $(\xi', \eta', \dots)$, &c. as coefficients, may be represented in the form

$$(\xi, \eta, \dots \check{\gamma}\, x, y, \dots), \qquad (\xi', \eta', \dots \check{\gamma}\, x', y', \dots), \dots$$

we may from the quantic (which for convenience I call $U$) form an operative quantic

$$(*\,\check{\gamma}\, \xi, \eta, \dots \overset{m}{\check{\gamma}}\, \xi', \eta', \dots \overset{m'}{)} \dots)$$

(I call this quantic $\Theta$), the coefficients of which are mere numerical multiples of $\partial_a, \partial_b, \partial_c, \dots$, and which is such that

$$\Theta U = \overline{(\xi, \eta, \dots \check{\gamma}\, x, y, \dots)}|^m\, \overline{(\xi', \eta', \dots \check{\gamma}\, x', y', \dots)}|^{m'} \dots$$

i.e. a product of powers of the linear functions. And it is to be remarked that as regards the quantic $\Theta$ and its covariants or other derivatives, the symbols $\partial_a, \partial_b, \partial_c, \dots$ are to be considered as elements with respect to which we may differentiate, &c.

The quantic $\Theta$ gives rise to the symbols $\{\xi\partial_\eta\}$, &c. analogous to the symbols $\{x\partial_y\}$, &c. formed from the quantic $U$. Suppose now that $\Phi$ is any quantic containing as well the coefficients as all or any of the sets of $\Theta$. Then $\{x\partial_y\}$ being a linear function of $a, b, c, \ldots$ the variables to which the differentiations in $\Phi$ relate, we have

$$\Phi \,.\, \{x\partial_y\} = \Phi\,\{x\partial_y\} + \Phi\,(\{x\partial_y\})\,;$$

again, $\{\iota\partial_\xi\}$ being a linear function of the differentiations with respect to the variables $\partial_a, \partial_b, \partial_c, \ldots$ in $\Phi$, we have

$$\{\eta\partial_\xi\} \,.\, \Phi = \{\eta\partial_\xi\}\,\Phi + \{\eta\partial_\xi\}\,(\Phi)\,;$$

these equations serve to show the meaning of the notations $\Phi\,(\{x\partial_y\})$ and $\{\eta\partial_\xi\}\,(\Phi)$, and there exists between these symbols the singular equation

$$\Phi\,(\{x\partial_y\}) = \{\eta\partial_\xi\}\,(\Phi).$$

14. The general demonstration of this equation presents no real difficulty, but to avoid the necessity of fixing upon a notation to distinguish the coefficients of the different terms and for the sake of simplicity, I shall merely exhibit by an example the principle of such general demonstration. Consider the quantic

$$U = ax^3 + 3bx^2y + 3cy^2 + dy^3,$$

this gives
$$\Theta = \xi^3\partial_a + \xi^2\eta\partial_b + \xi\eta^2\partial_c + \eta^3\partial_d\,;$$

or if, for greater clearness, $\partial_a, \partial_b, \partial_c, \partial_d$ are represented by $\alpha, \beta, \gamma, \delta$, then

$$\Theta = \alpha\xi^3 + \beta\xi^2\eta + \gamma\xi\eta^2 + \delta\eta^3,$$

and we have
$$\{x\partial_y\} = 3b\partial_a + 2c\partial_b + d\partial_c,$$

and
$$\{\eta\partial_\xi\} = 3\alpha\partial_\beta + 2\beta\partial_\gamma + \gamma\partial_\delta.$$

Now considering $\Phi$ as a function of $\partial_a, \partial_b, \partial_c, \partial_d$, or, what is the same thing, of $\alpha, \beta, \gamma, \delta$, we may write

$$\Phi\,(\{x\partial_y\}) = \Phi\,(3b\alpha + 2c\beta + d\gamma)\,;$$

and if in the expression of $\Phi$ we write $\alpha + \partial_a$, $\beta + \partial_b$, $\gamma + \partial_c$, $\delta + \partial_d$ for $\alpha, \beta, \gamma, \delta$ (where only the symbols $\partial_a, \partial_b, \partial_c, \partial_d$ are to be considered as affecting $a, b, c, d$ as contained in the operand $3b\alpha + 2c\beta + d\gamma$), and reject the first term (or term independent of $\partial_a, \partial_b, \partial_c, \partial_d$ in the expansion) we have the required value of $\Phi\,(\{x\delta_y\})$. This value is

$$(\partial_\alpha\,\Phi\,\partial_a + \partial_\beta\,\Phi\,\partial_b + \partial_\gamma\,\Phi\,\partial_c)\,(3b\alpha + 2c\beta + d\gamma)\,;$$

performing the differentiations $\partial_a, \partial_b, \partial_c, \partial_d$, the value is

$$(3\alpha\partial_\beta + 2\beta\partial_\gamma + \gamma\partial_\delta)\,\Phi,$$

i.e. we have
$$\Phi\,(\{x\partial_y\}) = \{\eta\partial_\xi\}\,(\Phi).$$

15. Suppose now that $\Phi$ is a covariant of $\Theta$, then the operation $\Phi$ performed upon any covariant of $U$ gives rise to a covariant of the system

$$(*\between x, y, \ldots \overset{m}{\between} x', y', \ldots \overset{m'}{)} \ldots),$$

$$(\xi, \eta, \ldots \between x, y, \ldots), \quad (\xi', \eta', \ldots \between x', y', \ldots), \text{ \&c.}$$

To prove this it is to be in the first instance noticed, that as regards $(\xi, \eta, \ldots \between x, y, \ldots)$, &c. we have $\{x\partial_y\} = \eta\partial_\xi$, &c. Hence considering $\{x\partial_y\}$, &c. as referring to the quantic $U$, the operation $\Sigma\{x\partial_y\} - x\partial_y$ will be equivalent to $\{x\partial_y\} + \eta\partial_\xi - x\partial_y$, and therefore every covariant of the system must be reduced to zero by each of the operations

$$\mathfrak{D} = \{x\partial_y\} + \eta\partial_\xi - x\partial_y.$$

This being the case, we have

$$\mathfrak{D} \,.\, \Phi = \mathfrak{D}\Phi + \mathfrak{D}(\Phi),$$
$$\Phi \,.\, \mathfrak{D} = \Phi\mathfrak{D} + \Phi(\mathfrak{D}),$$

equations which it is obvious may be replaced by

$$\mathfrak{D} \,.\, \Phi = \mathfrak{D}\Phi + \eta\partial_\xi(\Phi),$$
$$\Phi \,.\, \mathfrak{D} = \Phi\mathfrak{D} + \Phi(\{x\partial_y\}),$$

and consequently (in virtue of the theorem) by

$$\mathfrak{D} \,.\, \Phi = \mathfrak{D}\Phi + \eta\partial_\xi(\Phi),$$
$$\Phi \,.\, \mathfrak{D} = \Phi\mathfrak{D} + \{\eta\partial_\xi\}(\Phi);$$

and we have therefore

$$\mathfrak{D} \,.\, \Phi - \Phi \,.\, \mathfrak{D} = -(\{\eta\partial_\xi\} - \eta\partial_\xi)(\Phi);$$

or, since $\Phi$ is a covariant of $\Theta$, we have $\mathfrak{D} \,.\, \Phi = \Phi \,.\, \mathfrak{D}$. And since every covariant of the system is reduced to zero by the operation $\mathfrak{D}$, and therefore by the operation $\Phi \,.\, \mathfrak{D}$, such covariant will also be reduced to zero by the operation $\mathfrak{D} \,.\, \Phi$, or what is the same thing, the covariant operated on by $\Phi$, is reduced to zero by the operation $\mathfrak{D}$ and is therefore a covariant, i.e. $\Phi$ operating upon a covariant gives a covariant.

16. In the case of a quantic such as $U =$

$$(*\between x, y \overset{m}{\between} x', y' \overset{m'}{)} \ldots),$$

we may instead of the new sets $(\xi, \eta)$, $(\xi', \eta')$... employ the sets $(y, -x)$, $(y', -x')$, &c. The operative quantic $\Theta$ is in this case defined by the equation $\Theta U = 0$, and if $\Phi$ be, as before, any covariant of $\Theta$, then $\Phi$ operating upon a covariant of $U$ will give a covariant of $U$. The proof is nearly the same as in the preceding case; we have instead of the equation $\Phi(\{x\partial_y\}) = \{\eta\partial_\xi\}(\Phi)$ the analogous equation

$$\Phi(\{x\partial_y\}) = -\{x\partial_y\}(\Phi),$$

where on the left-hand side $\{x\partial_y\}$ refers to $U$, but on the right-hand side $\{x\partial_y\}$ refers to $\Theta$, and instead of $\mathfrak{D} = \{x\partial_y\} + \eta\partial_\xi - x\partial_y$ we have simply $\mathfrak{D} = \{x\partial_y\} - x\partial_y$.

17. I pass next to the quantic

$$(*\between x, y)^m,$$

which I shall in general consider under the form

$$(a, b, \ldots b', a' \between x, y)^m,$$

but sometimes under the form

$$(a, b, \ldots b', a' \between x, y)^m,$$

the former notation denoting, it will be remembered,

$$ax^m + \frac{m}{1} bx^{m-1} y \ldots + \frac{m}{1} b' xy^{m-1} + a' y^m,$$

and the latter notation

$$ax^m + bx^{m-1} y \ldots + b' xy^{m-1} + a' y^m.$$

But in particular cases the coefficients will be represented all of them by unaccentuated letters, thus $(a, b, c, d \between x, y)^3$ will be used to denote $ax^3 + 3bx^2y + 3cxy^2 + dy^3$, and $(a, b, c, d \between x, y)^3$ will be used to denote $ax^3 + bx^2y + cxy^2 + dy^3$, and so in all similar cases.

Applying the general methods to the quantic

$$(a, b, \ldots b', a' \between x, y)^m,$$

we see that

$$\{y\partial_x\} = a\partial_b + 2b\partial_c \ldots + mb'\partial_{a'},$$

$$\{x\partial_y\} = mb\partial_a + (m-1c\partial_b \ldots + a'\partial_{b'};$$

in fact, with these meanings of the symbols the quantic is reduced to zero by each of the operations $\{y\partial_x\} - y\partial_x$, $\{x\partial_y\} - x\partial_y$; hence according to the definition any function which is reduced to zero by each of the last-mentioned operations is a covariant of the quantic. But in accordance with a preceding remark, the covariant may be considered as a rational and integral function, separately homogeneous in regard to the facients $(x, y)$ and the coefficients $(a, b, \ldots b', a')$. If instead of the single set $(x, y)$ the covariant contains the sets $(x_1, y_1)$, $(x_2, y_2)$, &c., then it must be reduced to zero by each of the operations $\{y\partial_x\} - Sy\partial_x$, $\{x\partial_y\} - Sx\partial_y$ (where $Sy\partial_x = y_1\partial_{x_1} + y_2\partial_{x_2} + \ldots$), but I shall principally attend to the case in which the covariant contains only the set $(x, y)$.

Suppose, for shortness, that the quantic is represented by $U$, and let $U_1, U_2, \ldots$ be what $U$ becomes when the set $(x, y)$ is successively replaced by the sets $(x_1, y_1)$, $(x_2, y_2)$, &c. Suppose moreover that $\overline{12} = \partial_{x_1}\partial_{y_2} - \partial_{x_2}\partial_{y_1}$, &c., then the function

$$\overline{12}^p\, \overline{13}^q\, \overline{23}^r \ldots U_1 U_2 U_3 \ldots,$$

in which, after the differentiations, the new sets $(x_1, y_1)$, $(x_2, y_2)$, ... may be replaced by the original set $(x, y)$, will be a covariant of the quantic $U$. And if the number of differentiations be such as to make the facients disappear, i.e. if the sum of all the indices $p, q, \ldots$ of the terms $\overline{12}$, &c. which contain the symbolic number 1, the sum of all the indices $p, r, \ldots$ of the terms which contain the symbolic number 2, and so on, be severally equal to the degree of the quantic, we have an invariant. The operative quantic $\Theta$ becomes in the case under consideration

$$\Theta = (\partial_{a'}, -\partial_{b'}, \ldots \pm \partial_a \between x, y)^m,$$

the signs being alternately positive and negative; in fact it is easy to verify that this expression gives identically $\Theta U = 0$, and any covariant of $\Theta$ operating on a covariant of $U$ gives rise to a covariant of $U$.

18. But the quantic

$$(a, b, \ldots b', a' \between x, y)^m,$$

considered as decomposable into linear factors, i.e. as expressible in the form

$$a(x - \alpha y)(x - \beta y) \ldots,$$

gives rise to a fresh series of results. We have in this case

$$\{y\partial_x\} = \qquad \partial_\alpha + \partial_\beta \ldots,$$

$$\{x\partial_y\} = -(\alpha + \beta \ldots)\, a\partial_a + \alpha^2\partial_\alpha + \beta^2\partial_\beta + \ldots;$$

in fact with these meanings of the symbols the quantic is reduced to zero by each of the operations $\{x\partial_y\} - x\partial_y$, $\{y\partial_x\} - y\partial_x$, and we have consequently the definition of the covariant of a quantic considered as expressed in the form $a(x - \alpha y)(x - \beta y) \ldots$. And it will be remembered that these and the former values of the symbols $\{x\partial_y\}$ and $\{y\partial_x\}$ are, when the same quantic is considered as represented under the two forms $(a, b, \ldots b', a' \between x, y)^m$ and $a(x - \alpha y)(x - \beta y) \ldots$, identical.

19. Consider now the expression

$$a^\theta (x - \alpha y)^j (x - \beta y)^k \ldots (\alpha - \beta)^p \ldots,$$

where the sum of the indices $j, p, \ldots$ of all the simple factors which contain $\alpha$, the sum of the indices $k, p, \ldots$ of all the simple factors which contain $\beta$, &c. are respectively equal to the index $\theta$ of the coefficient $a$. The index $\theta$ and the indices $p$, &c. may be considered as arbitrary, nevertheless within such limits as will give positive values (0 inclusive) for the indices $j, k, \ldots$.

The expression in question is reduced to zero by each of the operations $\{x\partial_y\} - x\partial_y$, $\{y\partial_x\} - y\partial_x$; and this is of course also the case with the expressions obtained by interchanging in any manner the roots $\alpha, \beta, \gamma, \ldots$, and therefore with the expression

$$a^\theta\, \Sigma\, (x - \alpha y)^j (x - \beta y)^k \ldots (\alpha - \beta)^p \ldots,$$

where $\Sigma$ denotes a summation with respect to all the different permutations of the roots $\alpha, \beta \ldots$.

The function so obtained (which is of course a rational function of $(a, b, \dots b', a')$) will be a covariant, and if we suppose $\mu = m\theta - 2Sp$, where $Sp$ denotes the sum of all the indices $p$ of the different terms $(\alpha - \beta)^p$, &c., then the covariant will be of the order $\mu$ (i.e. of the degree $\mu$ in the facients $x, y$), and of the degree $\theta$ in the coefficients.

20. In connexion with this covariant

$$a^\theta \Sigma (x - \alpha y)^p (x - \beta y)^k \dots (\alpha - \beta)^p \dots,$$

of the order $\mu$ and of the degree $\theta$ in the coefficients, of the quantic $U =$

$$a (x - \alpha y)(x - \beta y) \dots,$$

consider the covariant

$$\Sigma (\overline{12}^p \dots) V_1 V_2 \dots V_m$$

of a quantic $V =$

$$(* \between x, y)^\phi,$$

in which, after the differentiations, the sets $(x_1, y_1)$, $(x_2, y_2)$, ... are replaced by the original set $(x, y)$. The last-mentioned covariant will be of the order $m(\phi - \theta) + \mu$, and will be of the degree $m$ in the coefficients; and in particular if $\phi = \theta$, i.e. if $V$ be a quantic of the order $\theta$, then the covariant will be of the order $\mu$ and of the degree $m$ in the coefficients. Hence to a covariant of the degree $\theta$ in the coefficients, of a quantic of the order $m$, there corresponds a covariant of the degree $m$ in the coefficients, of a quantic of the order $\theta$; the two covariants in question being each of them of the same order $\mu$. And it is proper to notice, that if we had commenced with the covariant of the quantic $V$, a reverse process would have led to the covariant of the quantic $U$. We may, therefore, say that the covariants of a given order and of the degree $\theta$ in the coefficients, of a quantic of the order $m$, correspond each to each with the covariants of the same order and of the degree $m$ in the coefficients, of a quantic of the order $\theta$; and in particular the invariants of the degree $\theta$ of a quantic of the order $m$, correspond each to each with the invariants of the degree $m$ of a quantic of the order $\theta$. This is the law of reciprocity demonstrated by M. Hermite, by a method which (I am inclined to think) is substantially identical with that here made use of, although presented in a very different form: the discovery of the law, considered as a law relating to the *number* of invariants, is due to Mr Sylvester. The precise meaning of the law, in the last-mentioned point of view, requires some explanation. Suppose that we know all the really independent invariants of a quantic of the order $m$, the law gives the number of invariants of the degree $m$ of a quantic of the order $\theta$ (it is convenient to assume $\theta > m$), viz. of the invariants of the degree in question, which are linearly independent, or asyzygetic, i.e. such that there do not exist any merely numerical multiples of these invariants having the sum zero; but the invariants in question may and in general will be connected *inter se* and with the other invariants of the quantic to which they belong by non-linear equations: and in particular the system of invariants of the degree $m$ will comprise all the invariants of that degree (if any) which are rational and integral functions of the invariants of lower degrees. The like observations apply to the system of covariants of a given order and of the degree $m$ in the coefficients, of a quantic of the order $\theta$.

21. The number of the really independent covariants of a quantic $(*\between x, y)^m$ is precisely equal to the order $m$ of the quantic, i.e. any covariant is a function (generally an irrational function only expressible as the root of an equation) of any $m$ independent covariants, and in like manner the number of really independent invariants is $m-2$; we may, if we please, take $m-2$ really independent invariants as part of the system of the $m$ independent covariants; the quantic itself may be taken as one of the other two covariants, and any other covariant as the other of the two covariants; we may therefore say that every covariant is a function (generally an irrational function only expressible as the root of an equation) of $m-2$ invariants, of the quantic itself and of a given covariant.

22. Consider any covariant of the quantic

$$(a, b, \ldots b', a' \between x, y)^m,$$

and let this be of the order $\mu$, and of the degree $\theta$ in the coefficients. It is very easily shown that $m\theta-\mu$ is necessarily even. In particular in the case of an invariant (i.e. when $\mu=0$) $m\theta$ is necessarily even[1]: so that a quantic of an odd order admits only of invariants of an even degree. But there is an important distinction between the cases of $m\theta-\mu$ evenly even and oddly even. In the former case the covariant remains unaltered by the substitution of $(y, x)$, $(a', b', \ldots b, a)$ for $(x, y)$, $(a, b, \ldots b', a')$; in the latter case the effect of the substitution is to change the sign of the covariant. The covariant may in the former case be called a symmetric covariant, and in the latter case a skew covariant. It may be noticed in passing, that the simplest skew invariant is M. Hermite's invariant of the degree 18 of a quantic of the order 5.

23. There is another very simple condition which is satisfied by every covariant of the quantic

$$(a, b, \ldots b', a' \between x, y)^m,$$

viz. if we consider the facients $(x, y)$ as being respectively of the weights $\frac{1}{2}, -\frac{1}{2}$, and the coefficients $(a, b, \ldots b', a')$ as being respectively of the weights $-\frac{1}{2}m, -\frac{1}{2}m+1, \ldots \frac{1}{2}m-1, \frac{1}{2}m$, then the weight of each term of the covariant will be zero. This is the most elegant statement of the law, but to avoid negative quantities, the statement may be modified as follows:—if the facients $(x, y)$ are considered as being of the weights 1, 0 respectively, and the coefficients $(a, b \ldots b', a')$ as being of the weights $0, 1, \ldots, m-1, m$ respectively, then the weight of each term of the covariant will be $\frac{1}{2}(m\theta+\mu)$.

[1] I may remark that it was only M. Hermite's important discovery of an invariant of the degree 18 of a quantic of the order 5, which removed an erroneous impression which I had been under from the commencement of the subject, that $m\theta$ was of necessity *evenly even*.

24. The preceding laws as to the form of a covariant have been stated here by way of anticipation, principally for the sake of the remark, that they so far define the form of a covariant as to render it in very many cases practicable with a moderate amount of labour to complete the investigations by means of the operation $\{x\partial_y\} - x\partial_y$ and $\{y\partial_x\} - y\partial_x$. In fact, for finding the covariants of a given order, and of a given degree in the coefficients, we may form the most general function of the proper order and degree in the coefficients, satisfying the prescribed conditions as to symmetry and weight: such function, if reduced to zero by one of the operations in question, will, on account of the symmetry, be reduced to zero by the other of the operations in question; it is therefore only necessary to effect upon it, e.g. the operation $\{x\partial_y\} - x\partial_y$, and to determine if possible the indeterminate coefficients in such manner as to render the result identically zero: of course when this cannot be done there is not any covariant of the form in question. It is moreover proper to remark, as regards invariants, that if an invariant be expanded in a series of ascending powers of the first coefficient $a$, and the first term of the expansion is known, all the remaining terms can be at once deduced by mere differentiations. There is one very important case in which the value of such first term (i.e. the value of the invariant when $a$ is put equal to 0) can be deduced from the corresponding invariant of a quantic of the next inferior order; the case in question is that of the discriminant (or function which equated to zero expresses the equality of a pair of roots); for by Joachimsthal's theorem, if in the discriminant of the quantic $(a, b, \dots b', a' \between x, y)^m$ we write $a = 0$, the result contains $b^2$ as a factor, and divested of this factor is precisely the discriminant of the quantic of the order $m - 1$ obtained from the given quantic by writing $a = 0$ and throwing out the factor $x$: this is in practice a very convenient method for the calculation of the discriminants of quantics of successive orders. It is also to be noticed as regards covariants, that when the first or last coefficient of any covariant (i.e. the coefficient of the highest power of either of the facients) is known, all the other coefficients can be deduced by mere differentiations.

Postscript added October 7th, 1854.—I have, since the preceding memoir was written, found with respect to the covariants of a quantic $(* \between x, y)^m$, that a function of any order and degree in the coefficients satisfying the necessary condition as to weight, and such that it is reduced to zero by one of the operations $\{x\partial_y\} - x\partial_y$, $\{y\partial_x\} - y\partial_x$, will of necessity be reduced to zero by the other of the two operations, i.e. it will be a covariant; and I have been thereby led to the discovery of the law for the number of asyzygetic covariants of a given order and degree in the coefficients; from this law I deduce as a corollary, the law of reciprocity of MM. Sylvester and Hermite. I hope to return to the subject in a subsequent memoir.