141.

A SECOND MEMOIR UPON QUANTICS.

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THE present memoir is intended as a continuation of my Introductory Memoir upon Quantics, t. CXLIV. (1854), p. 245, and must be read in connexion with it; the paragraphs of the two Memoirs are numbered continuously. The special subject of the present memoir is the theorem referred to in the Postscript to the Introductory Memoir, and the various developments arising thereout in relation to the number and form of the covariants of a binary quantic.

25. I have already spoken of asyzygetic covariants and invariants, and I shall have occasion to speak of irreducible covariants and invariants. Considering in general a function u determined like a covariant or invariant by means of a system of partial differential equations, it will be convenient to explain what is meant by an asyzygetic integral and by an irreducible integral. Attending for greater simplicity only to a single set $(a, b, c, \ldots)$, which in the case of the covariants or invariants of a single function will be as before the coefficients or elements of the function, it is assumed that the system admits of integrals of the form $u = P$, $u = Q$, &c., or as we may express it, of integrals P, Q, &c., where P, Q, &c. are rational and integral homogeneous functions of the set $(a, b, c, \ldots)$, and moreover that the system is such that P, Q, &c. being integrals, $\phi(P, Q, \ldots)$ is also an integral. Then considering only the integrals which are rational and integral homogeneous functions of the set $(a, b, c, \ldots)$, integrals $P, Q, R, \ldots$ not connected by any linear equation or syzygy (such as $\lambda P + \mu Q + \nu R \ldots 0$), [1] are said to be asyzygetic; but in speaking of the asyzygetic integrals of a particular degree, it is implied that the integrals are a system such that every other integral of

[1] It is hardly necessary to remark, that the multipliers λ, μ, ν, ..., and generally any coefficients or quantities not expressly stated to contain the set $(a, b, c, \ldots)$, are considered as independent of the set, or to use a convenient word, are considered as "trivials."

the same degree can be expressed as a linear function (such as $\lambda P + \mu Q + \nu R \ldots$) of these integrals; and any integral P not expressible as a rational and integral homogeneous function of integrals of inferior degrees is said to be an irreducible integral.

26. Suppose now that A_1, A_2, A_3, &c. denote the number of asyzygetic integrals of the degrees 1, 2, 3, &c. respectively, and let α_1, α_2, α_3, &c. be determined by the equations

$$\begin{aligned}
A_1 &= \alpha_1,\\
A_2 &= \tfrac{1}{2}\,\alpha_1(\alpha_1+1)+\alpha_2,\\
A_3 &= \tfrac{1}{6}\,\alpha_1(\alpha_1+1)(\alpha_1+2)+\alpha_1\alpha_2+\alpha_3,\\
A_4 &= \tfrac{1}{24}\,\alpha_1(\alpha_1+1)(\alpha_1+2)(\alpha_1+3)+\tfrac{1}{2}\,\alpha_1(\alpha_1+1)\,\alpha_2+\alpha_1\alpha_3+\tfrac{1}{2}\,\alpha_2(\alpha_2+1)+\alpha_4, \text{ \&c.},
\end{aligned}$$

or what is the same thing, suppose that

$$1 + A_1x + A_2x^2 + \text{\&c.} = (1-x)^{-\alpha_1}(1-x^2)^{-\alpha_2}(1-x^3)^{-\alpha_3}\ldots;$$

a little consideration will show that α_r represents the number of irreducible integrals of the degree r less the number of linear relations or syzygies between the composite or non-irreducible integrals of the same degree. In fact the asyzygetic integrals of the degree 1 are necessarily irreducible, i.e. $A_1 = \alpha_1$. Represent for a moment the irreducible integrals of the degree 1 by X, X', &c., then the composite integrals X^2, XX', &c., the number of which is $\frac{1}{2}\,\alpha_1(\alpha_1+1)$, must be included among the asyzygetic integrals of the degree 2; and if the composite integrals in question were asyzygetic, there would remain $A_2 - \frac{1}{2}\,\alpha_1(\alpha_1+1)$ for the number of irreducible integrals of the degree 2; but if there exist syzygies between the composite integrals in question, the number to be subtracted from A_2 will be $\frac{1}{2}\,\alpha_1(\alpha_1+1)$ less the number of these syzygies, and we shall have $A_2 - \frac{1}{2}\,\alpha_1(\alpha_1+1)$, i.e. α_2 equal to the number of the irreducible integrals of the degree 2 less the number of syzygies between the composite integrals of the same degree. Again, suppose that α_2 is negative $= -\beta_2$, we may for simplicity suppose that there are no irreducible integrals of the degree 2, but that the composite integrals of this degree, X^2, XX', &c., are connected by β_2 syzygies, such as $\lambda X^2 + \mu XX' + \text{\&c.} = 0$, $\lambda_1 X^2 + \mu_1 XX' + \text{\&c.} = 0$. The asyzygetic integrals of the degree 4 include X^4, X^3X', &c., the number of which is $\frac{1}{24}\,\alpha_1(\alpha_1+1)(\alpha_1+2)(\alpha_1+3)$; but these composite integrals are not asyzygetic, they are connected by syzygies which are augmentatives of the β_2 syzygies of the second degree, viz. by syzygies such as

$$(\lambda X^2 + \mu XX'\ldots)\,X^2 = 0, \quad (\lambda X^2 + \mu XX'\ldots)\,XX' = 0, \text{ \&c. } (\lambda_1 X^2 + \mu_1 XX'\ldots)\,X^2 = 0,$$

$$(\lambda_1 X^2 + \mu_1 XX'\ldots)\,XX' = 0, \text{ \&c.},$$

the number of which is $\frac{1}{2}\alpha_1(\alpha_1+1)\,\beta_2$. And these syzygies are themselves not asyzygetic, they are connected by secondary syzygies such as

$$\begin{aligned}
&\lambda_1(\lambda X^2 + \mu XX'\ldots)\,X^2 + \mu_1(\lambda X^2 + \mu XX'\ldots)\,XX' + \text{\&c.}\\
&\qquad - \lambda(\lambda_1 X^2 + \mu_1 XX'\ldots)\,X^2 - \mu(\lambda_1 X^2 + \mu_1 XX'\ldots)\,XX' - \text{\&c.} = 0, \text{ \&c. \&c.},
\end{aligned}$$

the number of which is $\frac{1}{2}\beta_2(\beta_2-1)$. The real number of syzygies between the composite integrals X^4, X^3X', &c. (i.e. of the syzygies arising out of the β_2 syzygies between X^2, XX', &c.), is therefore $\frac{1}{2}\alpha_1(\alpha_1+1)\beta_2-\frac{1}{2}\beta_2(\beta_2-1)$, and the number of integrals of the degree 4, arising out of the integrals and syzygies of the degrees 1 and 2 respectively, is therefore

$$\tfrac{1}{24}\alpha_1(\alpha_1+1)(\alpha_1+2)(\alpha_1+3)-\tfrac{1}{2}\alpha_1(\alpha_1+1)\beta_2+\tfrac{1}{2}\beta_2(\beta_2-1);$$

or writing $-\alpha_2$ instead of β_2, the number in question is

$$\tfrac{1}{24}\alpha_1(\alpha_1+1)(\alpha_1+2)(\alpha_1+3)+\tfrac{1}{2}\alpha_1(\alpha_1+1)\alpha_2+\tfrac{1}{2}\alpha_2(\alpha_2+1).$$

The integrals of the degrees 1 and 3 give rise to $\alpha_1\alpha_3$ integrals of the degree 4; and if all the composite integrals obtained as above were asyzygetic, we should have

$$A_4-\tfrac{1}{24}\alpha_1(\alpha_1+1)(\alpha_1+2)(\alpha_1+3)-\tfrac{1}{2}\alpha_1(\alpha_1+1)\alpha_2-\tfrac{1}{2}\alpha_2(\alpha_2+1)-\alpha_1\alpha_3,$$

i.e. α_4 as the number of irreducible integrals of the degree 4; but if there exist any further syzygies between the composite integrals, then α_4 will be the number of the irreducible integrals of the degree 4 less the number of such further syzygies, and the like reasoning is in all cases applicable.

27. It may be remarked, that for any given partial differential equation, or system of such equations, there will be always a finite number ν such that given ν independent integrals every other integral is a function (in general an irrational function only expressible as the root of an equation) of the ν independent integrals; and if to these integrals we join a single other integral not a rational function of the ν integrals, it is easy to see that every other integral will be a rational function of the $\nu+1$ integrals; but every such other integral will not in general be a rational and integral function of the $\nu+1$ integrals; and [*incorrect*] there is not in general any finite number whatever of integrals, such that every other integral is a rational and integral function of these integrals, i.e. the number of irreducible integrals is in general infinite; and it would seem that this is in fact the case in the theory of covariants.

28. In the case of the covariants, or the invariants of a binary quantic, A_n is given (this will appear in the sequel) as the coefficient of x^n in the development, in ascending powers of x, of a rational fraction $\dfrac{\phi x}{fx}$, where fx is of the form

$$(1-x)^{\beta_1}(1-x^2)^{\beta_2}\ldots(1-x^k)^{\beta_k},$$

and the degree of ϕx is less than that of fx. We have therefore

$$1+A_1x+A_2x^2+\ldots=\frac{\phi x}{fx},$$

and consequently

$$\phi x=(1-x)^{\beta_1-\alpha_1}(1-x^2)^{\beta_2-\alpha_2}\ldots(1-x^k)^{\beta_k-\alpha_k}(1-x^{k+1})^{-\alpha_{k+1}}\ldots$$

Now every rational factor of a binomial $1-x^m$ is the irreducible factor of $1-x^{m'}$, where m' is equal to or a submultiple of m. Hence in order that the series $\alpha_1, \alpha_2, \alpha_3, \ldots$ may terminate, ϕx must be made up of factors each of which is the irreducible factor of a binomial $1-x^m$, or if ϕx be itself irreducible, then ϕx must be the irreducible factor of a binomial $1-x^m$. Conversely, if ϕx be not of the form in question, the series $\alpha_1, \alpha_2, \alpha_3$, &c. will go on *ad infinitum*, and it is easy to see that there is no point in the series such that the terms beyond that point are all of them negative, i.e. there will be irreducible covariants or invariants of indefinitely high degrees; and the number of covariants or invariants will be infinite. The number of invariants is first infinite in the case of a quantic of the seventh order, or septimic; the number of covariants is first infinite in the case of a quantic of the fifth order, or quintic. [As is now well known, these conclusions are incorrect, the number of irreducible covariants or invariants is in every case finite.]

29. Resuming the theory of binary quantics, I consider the quantic

$$(a, b, \ldots b^{\backprime}, a^{\backprime} \between x, y)^m.$$

Here writing

$$\{y\partial_x\} = a\partial_b + 2b\partial_c \ldots \qquad + mb^{\backprime}\partial_{a^{\backprime}},\ = X,$$

$$\{x\partial_y\} = mb\partial_a + (m-1)\,c\partial_b \ldots + a^{\backprime}\partial_{b^{\backprime}},\ = Y,$$

any function which is reduced to zero by each of the operations $X - y\partial_x$, $Y - x\partial_y$ is a covariant of the quantic. But a covariant will always be considered as a rational and integral function separately homogeneous in regard to the facients (x, y) and to the coefficients $(a, b, \ldots b^{\backprime}, a^{\backprime})$. And the words order and degree will be taken to refer to the facients and to the coefficients respectively.

I commence by proving the theorem enunciated, No. 23. It follows at once from the definition, that the covariant is reduced to zero by the operation

$$\overline{X - y\partial_x}\,.\,\overline{Y - x\partial_y} - \overline{Y - x\partial_y}\,.\,\overline{X - y\partial_x},$$

which is equivalent to

$$X\,.\,Y - Y\,.\,X + y\partial_y - x\partial_x.$$

Now

$$X\,.\,Y = XY + X\,(Y)$$
$$Y\,.\,X = YX + Y(X),$$

where XY and YX are equivalent operations, and

$$X\,(Y) = \ 1ma\partial_a + 2\,(m-1)\,b\partial_b \ldots + m1b^{\backprime}\partial_{b^{\backprime}},$$

$$Y\,(X) = \qquad m1b\partial_b \ldots + 2\,(m-1)\,b^{\backprime}\partial_{b^{\backprime}} + 1ma^{\backprime}\partial_{a^{\backprime}},$$

whence

$$X\,(Y) - Y\,(X) = ma\partial_a + (m-2)\,b\partial_b \ldots - (m-2)\,b^{\backprime}\partial_{b^{\backprime}} - ma^{\backprime}\partial_{a^{\backprime}},\ = k \text{ suppose},$$

and the covariant is therefore reduced to zero by the operation

$$k + y\partial_y - x\partial_x.$$

Now as regards a term $a^\alpha b^\beta \ldots b^{\backprime\beta^{\backprime}} a^{\backprime\alpha^{\backprime}}\,.\,x^i y^j$, we have

$$k = m\alpha + (m-2)\,\beta \ldots,\ - (m-2)\,\beta^{\backprime} - m\alpha^{\backprime}$$

$$y\partial_y - x\partial_x = j - i\,;$$

and we see at once that for each term of the covariant we must have

$$m\alpha + (m-2)\beta \ldots - (m-2)\beta' - m\alpha' + j - i = 0,$$

i.e. if (x, y) are considered as being of the weights $\frac{1}{2}, -\frac{1}{2}$ respectively, and $(a, b, \ldots b', a')$ as being of the weights $-\frac{1}{2}m, -\frac{1}{2}m+1, \ldots \frac{1}{2}m-1, \frac{1}{2}m$ respectively, then the weight of each term of the covariant is zero.

But if (x, y) are considered as being of the weights 1, 0 respectively, and $(a, b, \ldots b', a')$ as being of the weights $0, 1, \ldots m-1, m$ respectively, then writing the equation under the form

$$m(\alpha + \beta \ldots + \beta' + \alpha') + j + i - 2(\beta + \ldots + \overline{m-1}\beta' + m\alpha' + i) = 0,$$

and supposing that the covariant is of the order μ and of the degree θ, each term of the covariant will be of the weight $\frac{1}{2}(m\theta + \mu)$.

I shall in the sequel consider the weight as reckoned in the last-mentioned manner. It is convenient to remark, that as regards any function of the coefficients of the degree θ and of the weight q, we have

$$X \,.\, Y - Y \,.\, X = m\theta - 2q.$$

30. Consider now a covariant

$$(A, B, \ldots B', A' \between x, y)^\mu$$

of the order μ and of the degree θ; the covariant is reduced to zero by each of the operations $X - y\partial_x$, $Y - x\partial_y$, and we are thus led to the systems of equations

$$XA = 0,$$
$$XB = \mu A,$$
$$XC = (\mu - 1) B,$$
$$\vdots$$
$$XB' = 2C',$$
$$XA' = B';$$

and

$$YA = B,$$
$$YB = 2C,$$
$$\vdots$$
$$YC' = (\mu - 1) B',$$
$$YB' = \mu A',$$
$$YA' = 0.$$

Conversely if these equations are satisfied the function will be a covariant.

I assume that A is a function of the degree θ and of the weight $\frac{1}{2}(m\theta - \mu)$, satisfying the condition

$$XA = 0;$$

and I represent by YA, Y^2A, Y^3A, &c. the results obtained by successive operations with Y upon the function A. The function Y^sA will be of the degree θ and of the weight $\frac{1}{2}(m\theta-\mu)+s$. And it is clear that in the series of terms YA, Y^2A, Y^3A, &c., we must at last come to a term which is equal to zero. In fact, since m is the greatest weight of any coefficient, the weight of Y^s is at most equal to $m\theta$, and therefore if $\frac{1}{2}(m\theta-\mu)+s>m\theta$, or $s>\frac{1}{2}(m\theta+\mu)$, we must have $Y^sA=0$.

Now writing for greater simplicity XY instead of $X.Y$, and so in similar cases, we have, as regards Y^sA,

$$XY-YX=\mu-2s.$$

Hence

$$(XY-YX)A=\mu A,$$

and consequently

$$XYA=YXA+\mu A=\mu A.$$

Similarly

$$(XY-YX)\,YA=(\mu-2)\,YA,$$

and therefore

$$XY^2A=YXYA+(\mu-2)\,YA$$

$$=\mu\,YA+(\mu-2)\,YA=2(\mu-1)\,YA.$$

And again,

$$(XY-YX)\,Y^2A=(\mu-4)\,Y^2A,$$

and therefore

$$XY^3A=YXY^2A+(\mu-4)\,Y^2A$$

$$=2\,(\mu-1)\,Y^2A+(\mu-4)\,Y^2A=3\,(\mu-2)\,Y^2A,$$

or generally

$$XY^sA=s\,(\mu-s+1)\,Y^sA.$$

Hence putting $s=\mu+1$, $\mu+2$, &c., we have

$$XY^{\mu+1}A=0,$$

$$XY^{\mu+2}A=-(\mu+2)\,1\,.\,Y^{\mu+1}A,$$

$$XY^{\mu+3}A=-(\mu+3)\,2\,.\,Y^{\mu+2}A,$$

&c.,

equations which show that

$$Y^{\mu+1}A=0;$$

for unless this be so, i.e. if $Y^{\mu+1}A\neq 0$, then from the second equation $XY^{\mu+2}A\neq 0$, and therefore $Y^{\mu+2}A\neq 0$, from the third equation $XY^{\mu+3}\neq 0$, and therefore $X^{\mu+3}A\neq 0$, and so on *ad infinitum*, i.e. we must have $Y^{\mu+1}A=0$.

31. The suppositions which have been made as to the function A, give therefore the equations

$$XA \;=0,$$

$$XYA=\mu A,$$

$$XY^2A=2(\mu-1)\,YA,$$

$$\vdots$$

$$XY^{\mu}A=\mu Y^{\mu+1}A,$$

$$Y^{\mu+1}A=0;$$

and if we now assume

$$B=YA,\quad C=\tfrac{1}{2}YB,\ldots\, A' =\frac{1}{\mu}\,YB',$$

the system becomes

$$XA=0,$$

$$XB=\mu A,$$

$$XC=(\mu-1)\,B,$$

$$\vdots$$

$$XA'=B',$$

$$YA'=0;$$

so that the entire system of equations which express that $(A,\ B\ldots B',\ A'\between x,\ y)^{\mu}$ is a covariant is satisfied; hence

THEOREM. Given a quantic $(a,\ b,\ldots b',\ a'\between x,\ y)^m$; if A be a function of the coefficients of the degree θ and of the weight $\tfrac{1}{2}(m\theta-\mu)$ satisfying the condition $XA=0$, and if $B,\ C,\ldots B',\ A'$ are determined by the equations

$$B=YA,\quad C=\tfrac{1}{2}YB,\ldots\, A'=\frac{1}{\mu}\,YB',$$

then will

$$(A,\ B,\ldots B',\ A\between x,\ y)^{\mu}$$

be a covariant.

In particular, a function A of the degree θ and of the weight $\tfrac{1}{2}m\theta$, satisfying the condition $XA=0$, will (also satisfy the equation $YA=0$ and will) be an invariant.

32. I take now for A the most general function of the coefficients, of the degree θ and of the weight $\tfrac{1}{2}(m\theta-\mu)$; then XA is a function of the degree θ and of the weight $\tfrac{1}{2}(m\theta-\mu)-1$, and the arbitrary coefficients in the function A are to be determined so that $XA=0$. The number of arbitrary coefficients is equal to the number of terms in A, and the number of the equations to be satisfied is equal to the number of terms in XA; hence the number of the arbitrary coefficients which remains indeterminate is equal to the number of terms in A less the number of terms in XA; and since the covariant is completely determined when the leading coefficient is known,

the difference in question is equal to the number of the asyzygetic covariants, i.e. the number of the asyzygetic covariants of the order μ and the degree θ is equal to the number of terms of the degree θ and weight $\frac{1}{2}(m\theta-\mu)$, less the number of terms of the degree θ and weight $\frac{1}{2}(m\theta-\mu)-1$.

33. I shall now give some instances of the calculation of covariants by the method just explained. It is very convenient for this purpose to commence by forming the literal parts by Arbogast's Method of Derivations: we thus form tables such as the following :—

a	b	c

a^2	ab	ac b^2	bc	b^2

a	b	c	d

a^2	ab	ac b^2	ad bc	bd c^2	cd	d^2

a^3	a^2b	a^2c ab^2	a^2d abc b^3	abd ac^2 b^2c	acd b^2d bc^2	ad^2 bcd c^3	bd^2 c^2d	cd^2	d^3

a^4	a^3b	a^3c a^2b^2	a^3d a^2bc ab^3	a^2bd a^2c^2 ab^2c b^4	a^2cd ab^2d abc^2 b^3c	a^2d^2 $abcd$ ac^3 b^3d b^2c^2	abd^2 ac^2d b^2cd bc^3	acd^2 b^2d^2 bc^3d c^4	ad^3 bcd^2 c^3d	bd^3 c^2d^2	cd^3	d^4

a	b	c	d	e

a^2	ab	ac b^2	ad bc	ae bd c^2	be cd	bd c^2	cd	d^2

a^3	a^2b	a^2c	a^2d	a^2e	abe	ace	ade	ae^2	be^2	ce^2	de^2	e^3
		ab^2	abc	abd	acd	ad^2	bce	bde	cde	d^2e		
			b^3	ac^2	b^2d	b^2e	bd^2	c^2e	d^3			
				b^2c	bc^2	bcd	c^2d	cd^2				
						c^3						

34. Thus in the case of a cubic $(a, b, c, d \between x, y)^3$, the tables show that there will be a single invariant of the degree 4. Represent this by

$$\begin{aligned} &A a^2d^2 \\ &+ B abcd \\ &+ C ac^3 \\ &+ D b^3d \\ &+ E b^2c^2, \end{aligned}$$

which is to be operated upon with $a\partial_b + 2b\partial_c + 3c\partial_d$. This gives

$+ B$		$+ 6A$	a^2cd
$+ 3D$	$+ 2B$		ab^2d
$+ 2E$	$+ 6C$	$+ 3B$	abc^2
	$+ 4E$	$+ 3D$	b^3c

i.e. $B + 6A = 0$, $3D + 2B = 0$, &c.; or putting $A = 1$, we find $B = -6$, $C = 4$, $D = 4$, $E = -3$, and the invariant is

$$\begin{aligned} &a^2d^2 \\ &- 6\, abcd \\ &+ 4\, ac^3 \\ &+ 4\, b^3d \\ &- 3\, b^2c^2. \end{aligned}$$

Again, there is a covariant of the order 3 and the degree 3. The coefficient of x^3 or leading coefficient is

$$\begin{aligned} &A\, a^2d \\ &+ B\, abc \\ &+ C\, b^3, \end{aligned}$$

which operated upon with $a\partial_b + 2b\partial_c + 3c\partial_d$, gives

$+ B$		$+ 3A$	a^2c
$+ 3C$	$+ 2B$		ab^2

i.e. $B+3A=0$, $3C+2B=0$; or putting $A=1$, we have $B=-3$, $C=2$, and the leading coefficient is

$$\begin{array}{l} a^2d \\ -3\,abc \\ +2\,b^3. \end{array}$$

The coefficient of x^2y is found by operating upon this with $(3b\partial_a+2c\partial_b+d\partial_c)$, this gives

$$\begin{array}{|c|c|c|l} \hline +6 & & -3 & abd \\ & -6 & & ac^2 \\ -9 & +12 & & b^2c \\ \hline \end{array}$$

i.e. the required coefficient of x^2y is

$$\begin{array}{l} 3\,abd \\ -6\,ac^2 \\ +3\,b^2c\,; \end{array}$$

and by operating upon this with $\frac{1}{2}(3b\partial_a+2c\partial_b+d\partial_c)$, we have for the coefficient of xy^2,

$$\begin{array}{|c|c|c|l} \hline & +3 & -6 & acd \\ +\frac{9}{2} & & +\frac{3}{2} & b^2d \\ -9 & +6 & & bc^2 \\ \hline \end{array}$$

i.e. the coefficient of xy^2 is

$$\begin{array}{l} -3\,acd \\ +6\,b^2d \\ -3\,bc^2. \end{array}$$

Finally, operating upon this with $\frac{1}{3}(3b\partial_a+2c\partial_b+d\partial_c)$, we have for the coefficient of y^3,

$$\begin{array}{|c|c|c|l} \hline & & -1 & ad^2 \\ -3 & +8 & -2 & bcd \\ & -2 & & c^3 \\ \hline \end{array}$$

i.e. the coefficient of y^3 is

$$\begin{array}{l} -\;ad^2 \\ +3\,bcd \\ -2\,c^3, \end{array}$$

and the covariant is

$$\left(\begin{array}{|ll|ll|ll|ll|} a^2d & +1 & abd & +3 & acd & -3 & ad^2 & -1 \\ abc & -3 & ac^2 & -6 & b^2d & +6 & bcd & +3 \\ b^3 & +2 & b^2c & +3 & bc^2 & -3 & c^3 & -2 \end{array}\right)(x,\ y)^3$$

[I now write the numerical coefficients after instead of before the literal terms.]

I have worked out the example in detail as a specimen of the most convenient method for the actual calculation of more complicated covariants[1].

35. The number of terms of the degree θ and of the weight q is obviously equal to the number of ways in which q can be made up as a sum of θ terms with the elements $(0, 1, 2, \dots m)$, a number which is equal to the coefficient of $x^q z^\theta$ in the development of

$$\frac{1}{(1-z)(1-xz)(1-x^2z)\dots(1-x^mz)};$$

and the number of the asyzygetic covariants of any particular degree for the quantic $(*\between x, y)^m$ can therefore be determined by means of this development. In the case of a cubic, for example, the function to be developed is

$$\frac{1}{(1-z)(1-xz)(1-x^2z)(1-x^3z)},$$

which is equal to

$$1 + z(1 + x + x^2 + x^3) + z^2(1 + x + 2x^2 + 2x^3 + 2x^4 + 2x^5 + x^6) + \&\text{c.},$$

where the coefficients are given by the following table; on account of the symmetry, the series of coefficients for each power of z is continued only to the middle term or middle of the series.

									1	(0)
								1	1	(1)
						1	1	2	2	(2)
					1	1	2	3	3	(3)
			1	1	2	3	4	4	5	(4)
		1	1	2	3	4	5	6	6	(5)
1	1	2	3	4	5	7	7	8	8	(6)

[1] Note added Feb. 7, 1856.—The following method for the calculation of an invariant or of the leading coefficient of a covariant, is easily seen to be identical in principle with that given in the text. Write down all the terms of the weight next inferior to that of the invariant or leading coefficient, and operate on each of these separately with the symbol

$$\text{ind. } b \,.\, \frac{b}{a} + 2 \text{ ind. } c \,.\, \frac{c}{b} + \dots (m-1) \text{ ind. } b' \,.\, \frac{b'}{a'},$$

where we are first to multiply by the fraction, rejecting negative powers, and then by the index of the proper letter in the term so obtained. Equating the results to zero, we obtain equations between the terms of the invariant or leading coefficient, and replacing in these equations each term by its numerical coefficient in the and from this, by subtracting from each coefficient the coefficient which immediately precedes it, we form the table:

									1	(0)
								1	0	(1)
						1	0	1	0	(2)
					1	0	1	1	0	(3)
			1	0	1	1	1	0	1	(4)
		1	0	1	1	1	1	1	0	(5)
1	0	1	1	1	1	2	0	1	0	(6)

The successive lines fix the number and character of the covariants of the degrees 0, 1, 2, 3, &c. The line (0), if this were to be interpreted, would show that there is a single covariant of the degree 0; this covariant is of course merely the absolute constant unity, and may be excluded. The line (1) shows that there is a single covariant of the degree 1, viz. a covariant of the order 3; this is the cubic itself, which I represent by U. The line (2) shows that there are two asyzygetic covariants of the degree 2, viz. one of the order 6, this is merely U^2, and one of the order 2, this I represent by H. The line (3) shows that there are three asyzygetic covariants of the degree 3, viz. one of the order 9, this is U^3; one of the order 5, this is UH, and one of the order 3, this I represent by Φ. The line (4) shows that there are five asyzygetic covariants of the degree 4, viz. one of the order 12, this is U^4; one of the order 8, this is U^2H; one of the order 6, this is H^2; and one of the order 0, i.e. an invariant, this I represent by ∇. The line (5) shows that there are six asyzygetic covariants of the degree 5, viz. one of the order 15, this is U^5; one of the order 11, this is U^3H; one of the order 9, this is $U^2\Phi$; one of the order 7, this is UH^2; one of the order 5, this is $H\Phi$; and one of the order 3, this is ∇U. The line (6) shows that there are 8 asyzygetic covariants of the degree 6, viz. one of the order 18, this is U^6; one of the

invariant or leading coefficient, we have the equations of connexion of these numerical coefficients. Thus, for the discriminant of a cubic, the terms of the next inferior weight are a^2cd, ab^2d, abc^2, b^3c, and operating on each of these separately with the symbol

$$\text{ind. } b\,.\,\frac{b}{a}+2\text{ ind. } c\,.\,\frac{c}{b}+3\text{ ind. } d\,.\,\frac{d}{c},$$

we find

$abcd$		$+6\,a^2d^2$
$3\,b^3d$	$+2\,abcd$	
$2\,b^2c^2$	$+6\,ac^3$	$+3\,abcd$
	$+4\,b^2c^2$	$+3\,b^3d$

and equating the horizontal lines to zero, and assuming $a^2d^2=1$, we have $a^2d^2=1$, $abcd=-6$, $ac^3=4$, $b^3d=4$, $b^2c^2=-3$, or the value of the discriminant is that given in the text.

order 14, this is U^4H; one of the order 12, this is $U^3\Phi$; one of the order 10, this is U^2H^2; one of the order 8, this is $UH\Phi$; two of the order 6 (i.e. the three covariants H^3, Φ^2 and ∇U^2 are not asyzygetic, but are connected by a single linear equation or syzygy), and one of the order 2, this is ∇H. We are thus led to the irreducible covariants U, H, Φ, ∇ connected by a linear equation or syzygy between H^3, Φ^2 and ∇U^2, and this is in fact the complete system of irreducible covariants; ∇ is therefore the only invariant.

36. The asyzygetic covariants are of the form $U^pH^q\nabla^r$, or else of the form $U^pH^q\nabla^r\Phi$; and since U, H, ∇ are of the degrees 1, 2, 4 respectively, and Φ is of the degree 3, the number of asyzygetic covariants of the degree m of the first form is equal to the coefficient of x^m in $1 \div (1-x)(1-x^2)(1-x^4)$, and the number of the asyzygetic covariants of the degree m of the second form is equal to the coefficient of x^m in $x^3 \div (1-x)(1-x^2)(1-x^4)$. Hence the total number of asyzygetic covariants is equal to the coefficient of x^m in $(1+x^3) \div (1-x)(1-x^2)(1-x^4)$, or what is the same thing, in

$$\frac{1-x^6}{(1-x)(1-x^2)(1-x^3)(1-x^4)};$$

and conversely, if this expression for the number of the asyzygetic covariants of the degree m were established independently, it would follow that the irreducible invariants were four in number, and of the degrees 1, 2, 3, 4 respectively, but connected by an equation of the degree 6. As regards the invariants, every invariant is of the form ∇^p, i.e. the number of asyzygetic invariants of the degree m is equal to the coefficient of x^m in $\dfrac{1}{1-x^4}$, and conversely, from this expression it would follow that there was a single irreducible invariant of the degree 4.

37. In the case of a quartic, the function to be developed is:

$$\frac{1}{(1-z)(1-xz)(1-x^2z)(1-x^3z)(1-x^4z)},$$

and the coefficients are given by the table.

												1	(0)
										1	1	1	(1)
								1	1	2	2	3	(2)
						1	1	2	3	4	4	5	(3)
				1	1	2	3	5	5	7	7	8	(4)
		1	1	2	3	5	6	8	9	11	11	12	(5)
1	1	2	3	5	6	9	10	13	14	16	16	18	(6)

and subtracting from each coefficient the coefficient immediately preceding it, we have the table:

												1	(0)
										1	0	0	(1)
								1	0	1	0	1	(2)
						1	0	1	1	1	0	1	(3)
				1	0	1	1	2	0	2	0	1	(4)
		1	0	1	1	2	1	2	1	2	0	1	(5)
1	0	1	1	2	1	3	1	3	1	2	0	2	(6)

the examination of which will show that we have for the quartic the following irreducible covariants, viz. the quartic itself U; an invariant of the degree 2, which I represent by I; a covariant of the order 4 and of the degree 2, which I represent by H; an invariant of the degree 3, which I represent by J; and a covariant of the order 6 and the degree 3, which I represent by Φ; but that the irreducible covariants are connected by an equation of the degree 6, viz. there is a linear equation or syzygy between Φ^2, I^3H^3, I^2JH^2U, IJ^2HU^2 and J^3U^3; this is in fact the complete system of the irreducible covariants of the quartic: the only irreducible invariants are the invariants I, J.

38. The asyzygetic covariants are of the form $U^pI^qH^rJ^s$, or else of the form $U^pI^qH^rJ^s\Phi$, and the number of the asyzygetic covariants of the degree m is equal to the coefficient of x^m in $(1+x^3)\div(1-x)(1-x^2)^2(1-x^3)$, or what is the same thing, in

$$\frac{1-x^6}{(1-x)(1-x^2)^2(1-x^3)^2},$$

and the asyzygetic invariants are of the form I^pJ^q, and the number of the asyzygetic invariants of the degree m is equal to the coefficient of x^m in $1\div(1-x^2)(1-x^3)$. Conversely, if these formulæ were established, the preceding results as to the form of the system of the irreducible covariants or of the irreducible invariants, would follow.

39. In the case of a quintic, the function to be developed is

$$\frac{1}{(1-z)(1-xz)(1-x^2z)(1-x^3z)(1-x^4z)(1-x^5z)};$$

and the coefficients are given by the table:

												1	(0)
										1	1	1	(1)
							1	1	2	2	3	3	(2)
					1	1	2	3	4	5	6	6	(3)
		1	1	2	3	5	6	8	9	11	11	12	(4)
1	1	2	3	5	7	9	11	14	16	18	19	20	(5)
													⋮

and subtracting from each coefficient the one which immediately precedes it, we have the table:

												1	(0)
										1	0	0	(1)
							1	0	1	0	1	0	(2)
					1	0	1	1	1	1	1	0	(3)
		1	0	1	1	2	1	2	1	2	0	1	(4)
1	0	1	1	2	2	2	2	3	2	2	1	1	(5)
													⋮

We thus obtain the following irreducible covariants, viz.:

Of the degree 1; a single covariant of the order 5, this is the quintic itself.

Of the degree 2; two covariants, viz. one of the order 6, and one of the order 2.

Of the degree 3; three covariants, viz. one of the order 9, one of the order 5, and one of the order 3.

Of the degree 4; three covariants, viz. one of the order 6, one of the order 4, and one of the order 0 (an invariant).

Of the degree 5; three covariants, viz. one of the order 7, one of the order 3, and one of the order 1 (a linear covariant).

These covariants are connected by a single syzygy of the degree 5 and of the order 11; in fact, the table shows that there are only two asyzygetic covariants of this degree and order; but we may, with the above-mentioned irreducible covariants of the degrees, 1, 2, 3 and 4, form three covariants of the degree 5 and the order 11; there is therefore a syzygy of this degree and order.

40. I represent the number of ways in which q can be made up as a sum of m terms with the elements $0, 1, 2, \ldots m$, each element being repeatable an indefinite number of times by the notation

$$P(0, 1, 2, \ldots m)^\theta q,$$

and I write for shortness

$$P'(0, 1, 2, \ldots m)^\theta q = P(0, 1, 2 \ldots m)^\theta q - P(0, 1, 2 \ldots m)^\theta (q-1).$$

Then for a quantic of the order m, the number of asyzygetic covariants of the degree θ and of the order μ is

$$P'(0, 1, 2 \ldots m)^\theta \tfrac{1}{2}(m\theta - \mu).$$

In particular, the number of asyzygetic invariants of the degree θ is

$$P'(0, 1, 2 \ldots m)^\theta \tfrac{1}{2}m\theta.$$

To find the total number of the asyzygetic covariants of the degree θ, suppose first that $m\theta$ is even; then, giving to μ the successive values $0, 2, 4, \ldots m\theta$, the required number is

$$\begin{aligned}
&P(\tfrac{1}{2}m\theta) \quad\;\; - P(\tfrac{1}{2}m\theta - 1)\\
&+ P(\tfrac{1}{2}m\theta - 1) - P(\tfrac{1}{2}m\theta - 2)\\
&\quad\vdots\\
&+ P(2) \qquad\;\; - P(1)\\
&+ P(1)\\
&= P(\tfrac{1}{2}m\theta),
\end{aligned}$$

i.e. when $m\theta$ is even, the number of the asyzygetic covariants of the degree θ is

$$P(0, 1, 2 \ldots m)^\theta \tfrac{1}{2}m\theta;$$

and similarly, when $m\theta$ is odd, the number of the asyzygetic covariants of the degree θ is

$$P(0, 1, 2, \ldots m)^\theta \tfrac{1}{2}(m\theta - 1).$$

But the two formulæ may be united into a single formula; for when $m\theta$ is odd $\frac{1}{2}m\theta$ is a fraction, and therefore $P(\frac{1}{2}m\theta)$ vanishes, and so when $m\theta$ is even $\frac{1}{2}(m\theta-1)$ is a fraction, and $P\frac{1}{2}(m\theta - 1)$ vanishes; we have thus the theorem, that for a quantic of the order m:

The number of the asyzygetic covariants of the degree θ is

$$P(0, 1, 2 \ldots m)^\theta \tfrac{1}{2}m\theta + P(0, 1, 2, \ldots m)^\theta \tfrac{1}{2}(m\theta - 1).$$

41. The functions $P(\frac{1}{2}m\theta)$, &c. may, by the method explained in my "Researches on the Partition of Numbers," [140], be determined as the coefficients of x^θ in certain functions of x; I have calculated the following particular cases:—

Putting, for shortness,

$$P'(0, 1, 2, \ldots m)^\theta \tfrac{1}{2}m\theta = \text{coefficient } x^\theta \text{ in } \phi m,$$

then $$\phi 2 = \frac{1}{1-x^2},$$

$$\phi 3 = \frac{1}{1-x^4},$$

$$\phi 4 = \frac{1}{(1-x^2)(1-x^3)},$$

$$\phi 5 = \frac{1-x^6+x^{12}}{(1-x^4)(1-x^6)(1-x^8)},$$

$$\phi 6 = \frac{(1-x)(1+x-x^3-x^4-x^5+x^7+x^8)}{(1-x^2)^2(1-x^3)(1-x^4)(1-x^5)},$$

$$\phi 7 = \frac{1-x^6+2x^8-x^{10}+5x^{12}+2x^{14}+6x^{16}+2x^{18}+5x^{20}-x^{22}+2x^{24}-x^{26}+x^{32}}{(1-x^4)(1-x^6)(1-x^8)(1-x^{10})(1-x^{12})},$$

$$\phi 8 = \frac{(1-x)(1+x-x^3-x^4+x^6+x^7+x^8+x^9+x^{10}-x^{13}+x^{15}+x^{16})}{(1-x^2)^2(1-x^3)^2(1-x^4)(1-x^5)(1-x^7)};$$

$$P(0, 1, 2, \ldots m)^\theta \tfrac{1}{2}m\theta = \text{coefficient of } x^\theta \text{ in } \psi m,$$

then $$\psi 2 = \frac{1}{(1-x)(1-x^2)},$$

$$\psi 3 = \frac{1+x^4}{(1-x^2)^2(1-x^4)},$$

$$\psi 4 = \frac{1-x+x^2}{(1-x)^2(1-x^2)(1-x^3)},$$

$$\psi 5 = \frac{1+x^2+6x^4+9x^6+12x^8+9x^{10}+6x^{12}+x^{14}+x^{16}}{(1-x^2)^2(1-x^4)(1-x^6)(1-x^8)};$$

$$P(0, 1, 2, \ldots m)^\theta \tfrac{1}{2}(m\theta-1) = \text{coefficient of } x^\theta \text{ in } \psi_{\prime} m,$$

then $$\psi_{\prime} 3 = \frac{x+x^3}{(1-x^2)^2(1-x^4)},$$

$$\psi_{\prime} 5 = \frac{x+4x^3+8x^5+10x^7-10x^9+8x^{11}+4x^{13}+x^{15}}{(1-x^2)^2(1-x^4)(1-x^6)(1-x^8)}.$$

And from what has preceded, it appears that for a quantic of the order m, the number of asyzygetic covariants of the degree θ is for m even, coefficient x^θ in ψm, and for m odd, coefficient x^θ in $(\psi m + \psi_{\prime} m)$; and that the number of asyzygetic invariants of the degree θ is coefficient x^θ in ϕm. Attending first to the invariants:

42. For a quadric, the number of asyzygetic invariants of the degree θ is

$$\text{coefficient } x^\theta \text{ in } \frac{1}{1-x^2},$$

which leads to the conclusion that there is a single irreducible invariant of the degree 2.

43. For a cubic, the number of asyzygetic invariants of the degree θ is

$$\text{coefficient } x^\theta \text{ in } \frac{1}{1-x^4},$$

i.e. there is a single irreducible invariant of the degree 4.

44. For a quartic, the number of asyzygetic invariants of the degree θ is

$$\text{coefficient } x^\theta \text{ in } \frac{1}{(1-x^2)(1-x^3)},$$

i.e. there are two irreducible invariants of the degrees 2 and 3 respectively.

45. For a quintic, the number of asyzygetic invariants of the degree θ is

$$\text{coefficient } x^\theta \text{ in } \frac{1-x^6+x^{12}}{(1-x^4)(1-x^6)(1-x^8)}.$$

The numerator is the irreducible factor of $1-x^{36}$, i.e. it is equal to $(1-x^{36})(1-x^6) \div (1-x^{18})(1-x^{12})$; and substituting this value, the number becomes

$$\text{coefficient } x^\theta \text{ in } \frac{1-x^{36}}{(1-x^4)(1-x^8)(1-x^{12})(1-x^{18})},$$

i.e. there are in all four irreducible invariants, which are of the degrees 4, 8, 12 and 18 respectively; but these are connected by an equation of the degree 36, i.e. the square of the invariant of the degree 18 is a rational and integral function of the other three invariants; a result, the discovery of which is due to M. Hermite.

46. For a sextic, the number of asyzygetic invariants of the degree θ is

$$\text{coefficient } x^\theta \text{ in } \frac{(1-x)(1+x-x^3-x^4-x^5+x^7+x^8)}{(1-x^2)^2(1-x^3)(1-x^4)(1-x^5)};$$

the second factor of the numerator is the irreducible factor $1-x^{30}$, i.e. it is equal to $(1-x^{30})(1-x^5)(1-x^3)(1-x^2) \div (1-x^{15})(1-x^{10})(1-x^6)(1-x)$; and substituting this value, the number becomes

$$\text{coefficient } x^{30} \text{ in } \frac{1-x^{30}}{(1-x^2)(1-x^4)(1-x^6)(1-x^{10})(1-x^{15})},$$

i.e. there are in all five irreducible invariants, which are of the degrees 2, 4, 6, 10 and 15 respectively; but these are connected by an equation of the degree 30, i.e. the square of the invariant of the degree 15 is a rational and integral function of the other four invariants.

47. For a septimic, the number of asyzygetic invariants of the degree θ is

$$\text{coefficient } x^\theta \text{ in } \frac{1-x^6+2x^8-x^{10}+5x^{12}+2x^{14}+6x^{16}+2x^{18}+5x^{20}-x^{22}+2x^{24}-x^{26}+x^{32}}{(1-x^4)(1-x^6)(1-x^8)(1-x^{10})(1-x^{12})},$$

the numerator is equal to

$$(1-x^6)(1-x^8)^{-2}(1-x^{10})(1-x^{12})^{-5}(1-x^{14})^{-4}\ldots,$$

where the series of factors does not terminate; hence [*incorrect*, see p. 253] the number of irreducible invariants is infinite; substituting the preceding value, the number of asyzygetic invariants of the degree θ is

$$\text{coefficient } x^\theta \text{ in } (1-x^4)^{-1}(1-x^8)^{-3}(1-x^{12})^{-6}(1-x^{14})^{-4}\ldots$$

The first four indices give the number of irreducible invariants of the corresponding degrees, i.e. there are 1, 3, 6 and 4 irreducible invariants of the degrees 4, 8, 12 and 14 respectively, but there is no reason to believe that the same thing holds with respect to the indices of the subsequent terms. To verify this it is to be remarked, that there are 1, 4, 10 and 4 asyzygetic invariants of the degrees in question respectively; there is therefore one irreducible invariant of the degree 4; calling this X_4, there is only one composite invariant of the degree 8, viz. $X_4{}^2$; there are therefore three irreducible invariants of this degree, say X_8, X_8', X_8''. The composite invariants of the degree 12 are four in number, viz. X_4^3, X_4X_8, X_4X_8', X_4X_8'', *and these cannot be connected by any syzygy*, for if they were so, X_4^2, X_8, X_8', X_8'' would be connected by a syzygy, or there would be less than 3 irreducible invariants of the degree 8. Hence there are precisely 6 irreducible invariants of the degree 12. And since the irreducible invariants of the degrees 4, 8 and 12 do not give rise to any composite invariant of the degree 14, there are precisely 4 irreducible invariants of the degree 14.

48. For an octavic, the number of the asyzygetic invariants of the degree θ is

$$\text{coefficient } x^\theta \text{ in } \frac{(1-x)(1+x-x^3-x^4+x^6+x^7+x^8+x^9+x^{10}-x^{12}-x^{13}+x^{15}+x^{16})}{(1-x^2)^2(1-x^3)^2(1-x^4)(1-x^5)(1-x^7)};$$

and the second factor of the numerator is

$$(1-x)^{-1}(1-x^2)(1-x^3)^{-1}(1-x^6)^{-1}(1-x^8)^{-1}(1-x^9)^{-1}(1-x^{10})^{-1}(1-x^{16})(1-x^{17})(1-x^{18})\ldots,$$

where the series of factors does not terminate, hence [*incorrect*] the number of irreducible invariants is infinite. Substituting the preceding value, the number of the asyzygetic invariants of the degree θ is

$$\text{coeff. } x^\theta \text{ in } (1-x^2)^{-1}(1-x^3)^{-1}(1-x^4)^{-1}(1-x^5)^{-1}(1-x^6)^{-1}(1-x^7)^{-1}(1-x^8)^{-1}(1-x^9)^{-1}(1-x^{10})^{-1}(1-x^{16})(1-x^{17})(1-x^{18})\ldots$$

There is certainly one, and only one irreducible invariant for each of the degrees 2, 3, 4, 5 and 6 respectively; but the formula does not show the number of the irreducible invariants of the degrees 7, &c.; in fact, representing the irreducible invariants of the degrees 2, 3, 4, 5 and 6 by X_2, X_3, X_4, X_5, X_6, these give rise to 3 composite invariants of the degree 7, viz. $X_2X_2X_3$, X_2X_5, X_3X_4, which may or may not be connected by a syzygy; if they are not connected by a syzygy, there will be a single irreducible invariant of the degree 7; but if they are connected by a syzygy, there will be two irreducible invariants of the degree 7; it is useless at present to pursue the discussion further.

Considering next the covariants,—

49. For a quadric, the number of asyzygetic covariants of the degree θ is

$$\text{coefficient } x^\theta \text{ in } \frac{1}{(1-x)(1-x^2)},$$

i.e. there are two irreducible covariants of the degrees 1 and 2 respectively; these are of course the quadric itself and the invariant.

50. For a cubic, the number of the asyzygetic covariants of the degree θ is

$$\text{coefficient } x^\theta \text{ in } \frac{(1+x)(1+x^2)}{(1-x^2)^2(1-x^4)}.$$

The first factor of the numerator is the irreducible factor of

$$1-x^2, = (1-x^2) \div (1-x),$$

and the second factor of the numerator is the irreducible factor of

$$1-x^4, = (1-x^4) \div (1-x^2);$$

substituting these values, the number is

$$\text{coefficient } x^\theta \text{ in } \frac{1-x^6}{(1-x)(1-x^2)(1-x^3)(1-x^4)},$$

i.e. there are 4 irreducible covariants of the degrees 1, 2, 3, 4 respectively; but these are connected by an equation of the degree 6; the covariant of the degree 1 is the cubic itself U, the other covariants are the covariants already spoken of and represented by the letters H, Φ and ∇ respectively (H is of the degree 2 and the order 3, Φ of the degree 3 and the order 3, and ∇ is of the degree 4 and the order 0, i.e. it is an invariant).

51. For a quartic, the number of the asyzygetic covariants of the degree θ is

$$\text{coefficient } x^\theta \text{ in } \frac{1-x+x^2}{(1-x)^2(1-x^2)(1-x^3)},$$

the numerator of which is the irreducible factor of $1-x^6$, i.e. it is equal to $(1-x^6)(1-x) \div (1-x^2)(1-x^3)$. Making this substitution, the number is

$$\text{coefficient } x^\theta \text{ in } \frac{1-x^6}{(1-x)(1-x^2)^2(1-x^3)^2},$$

i.e. there are five irreducible covariants, one of the degree 1, two of the degree 2, and two of the degree 3, but these are connected by an equation of the degree 6. The irreducible covariant of the degree 1 is of course the quartic itself U, the other irreducible covariants are those already spoken of and represented by I, H, J, Φ respectively (I is of the degree 2 and the order 0, and J is of the degree 3 and the order 0, i.e. I and J are invariants, H is of the degree 2 and the order 4, Φ is of the degree 3 and the order 6).

52. For a quintic, the number of irreducible covariants of the degree θ is

$$\text{coeff. } x^\theta \text{ in } \frac{1+x+x^2+4x^3+6x^4+8x^5+9x^6+10x^7+12x^8+10x^9+9x^{10}+8x^{11}+6x^{12}+4x^{13}+x^{14}+x^{15}+x^{16}}{(1-x^2)^2(1-x^4)(1-x^6)(1-x^8)},$$

the numerator of which is

$$(1+x)^2(1-x+2x^2+x^3+2x^4+3x^5+x^6+5x^7+x^8+3x^9+2x^{10}+x^{11}+2x^{12}-x^{13}+x^{14});$$

the first factor is $(1-x)^{-2}(1-x^2)^2$, the second factor is

$$(1-x)(1-x^2)^{-2}(1-x^3)^{-3}(1-x^4)^{-2}(1-x^5)^{-2}(1-x^6)^5(1-x^7)^5(1-x^8)^7(1-x^9)^1(1-x^{10})^{-9}(1-x^{11})^{-19}\ldots,$$

which does not terminate; hence [*incorrect*] the number of irreducible covariants is infinite. Substituting the preceding values, the expression for the number of the asyzygetic covariants of the degree θ is

$$\text{coeff. } x^\theta \text{ in } (1-x)^{-1}(1-x^2)^{-2}(1-x^3)^{-3}(1-x^4)^{-3}(1-x^5)^{-2}(1-x^6)^4(1-x^7)^5(1-x^8)^6(1-x^9)^1(1-x^{10})^{-9}(1-x^{11})^{-19}\ldots,$$

which agrees with a previous result: the numbers of irreducible covariants for the degrees 1, 2, 3, 4 are 1, 2, 3 and 3 respectively, and for the degree 5, the number of irreducible covariants is three, but there is one syzygy between the composite covariants of the degree in question; the difference $3-1=2$ is the index taken with its sign reversed of the factor $(1-x^5)^{-2}$.

53. I consider a system of the asyzygetic covariants of any particular degree and order of a given quantic, the system may of course be replaced by a system the terms of which are any linear functions of those of the original system, and it is necessary to inquire what covariants ought to be selected as most proper to represent the system of asyzygetic covariants; the following considerations seem to me to furnish a convenient rule of selection. Let the literal parts of the terms which enter into the coefficients of the highest power of x or leading coefficients be represented by M_α, M_β, M_γ,... these quantities being arranged in the natural or alphabetical order; the first in order of these quantities M, which enters into the leading coefficient of a particular covariant, may for shortness be called the leading term of such covariant, and a covariant the leading term of which is posterior in order to the leading term of another covariant, may be said to have a lower leading term.

It is clear, that by properly determining the multipliers of the linear functions we may form a covariant the leading term of which is lower than the leading term of any other covariant (the definition implies that there is but one such covariant); call this Θ. We may in like manner form a covariant such that its leading term is lower than the leading term of every other covariant except Θ_1; or rather we may form a system of such covariants, since if Φ_2 be a covariant having the property in question, $\Phi_2+k\Theta_1$ will have the same property, but k may be determined so that the covariant shall not contain the leading term of Θ_1, i.e. we may form a covariant Θ_2 such that its leading term is lower than the leading term of every other covariant excepting Θ_1, and that the leading term of Θ_1 does not enter into Θ_2; and there is but one such covariant, Θ_2. Again, we may form a covariant Θ_3 such that its leading term is lower than the leading term of every other covariant excepting Θ_1 and Θ_2, and that the leading terms of Θ_1 and Θ_2 do not either of them enter into Θ_3; and there is but one such covariant, Θ_3. And so on, until we arrive at a covariant the leading term of which is higher than the leading terms of the other covariants, and which does not contain the leading terms of the other covariants. We have thus a series of covariants Θ_1, Θ_2, Θ_3, &c. containing the proper number of terms, and which covariants may be taken to represent the asyzygetic covariants of the degree and order in question.

In order to render the covariants Θ definite as well numerically as in regard to sign, we may suppose that the covariant is in its least terms (i.e. we may reject numerical factors common to all the terms), and we may make the leading term positive. The leading term with the proper numerical coefficient (if different from unity) and with the proper power of x (or the order of the function) annexed, will, when the covariants of a quantic are tabulated, be sufficient to indicate, without any ambiguity whatever, the particular covariant referred to. I subjoin a table of the covariants of a quadric, a cubic and a quartic, and of the covariants of the degrees 1, 2, 3, 4 and 5 respectively of a quintic, and also two other invariants of a quintic.

[Except for the quantic itself, the algebraical sum of the numerical coefficients in any column is $=0$, viz. the sum of the coefficients with the sign $+$ is equal to that of the coefficients with the sign $-$, and I have as a numerical verification inserted at the foot of each column this sum with the sign $\pm$].

Covariant Tables (Nos. 1 to 26).

No. 1.

$$\left(\begin{array}{|c|c|c|}\hline a+1 & b+2 & c+1 \\ \hline \end{array} \between x, y\right)^2.$$

No. 2.

$$\begin{array}{|l|}\hline ac+1 \\ b^2-1 \\ \hline \pm 1 \end{array}$$

The tables Nos. 1 and 2 are the covariants of a binary quadric. No. 1 is the quadric itself; No. 2 is the quadrinvariant, which is also the discriminant.

No. 3.

$$\left(\begin{array}{|c|c|c|c|}\hline a+1 & b+3 & c+3 & d+1 \\ \hline \end{array} \between x, y\right)^3.$$

No. 4.

$$\left(\begin{array}{|l|l|l|}\hline ac+1 & ad+1 & bd+1 \\ b^2-1 & bc-1 & c^2-1 \\ \hline \pm 1 & \pm 1 & \pm 1 \end{array} \between x, y\right)^2$$

No. 5.

$$\left(\begin{array}{|l|l|l|l|}\hline a^2d+1 & abd+3 & acd-3 & ad^2-1 \\ abc-3 & ac^2-6 & b^2d+6 & bcd+3 \\ b^3+2 & b^2c+3 & bc^2-3 & c^3-2 \\ \hline \pm 3 & \pm 6 & \pm 6 & \pm 3 \end{array} \between x, y\right)^3.$$

No. 6.

$$\begin{array}{|l|}\hline a^2d^2+1 \\ abcd-6 \\ ac^3+4 \\ b^3d+4 \\ b^2c^2-3 \\ \hline \pm 9 \end{array}$$

The tables Nos. 3, 4, 5 and 6 are the covariants of a binary cubic. No. 3 is the cubic itself; No. 4 is the quadricovariant, or Hessian; No. 5 is the cubicovariant; No. 6 is the invariant, or discriminant. And if we write No. $3=U$, No. $4=H$, No. $5=\Phi$, No. $6=\nabla$, then identically, $\Phi^2 - \nabla U^2 + 4H^3 = 0.$

No. 7.

$$(\;\begin{array}{|c|c|c|c|c|}\hline a+1 & b+4 & c+6 & d+4 & e+1 \\ \hline\end{array}\;)\!(x, y)^4$$

No. 8.

$$\begin{array}{|c|}\hline ae+1 \\ bd-4 \\ c^2+3 \\ \\ \hline \pm 4\end{array}$$

No. 9.

$$(\;\begin{array}{|c|c|c|c|c|}\hline ac+1 & ad+2 & ae+1 & be+2 & bd+1 \\ b^2-1 & bc-2 & bd+2 & cd-2 & c^2-1 \\ & & c^2-3 & & \\ \hline \pm 1 & \pm 2 & \pm 3 & \pm 2 & \pm 1\end{array}\;)\!(x, y)^4$$

No. 10.

$$\begin{array}{|c|}\hline ace+1 \\ ad^2-1 \\ b^2e-1 \\ bcd+2 \\ c^3-1 \\ \hline \pm 3\end{array}$$

No. 11.

$$(\;\begin{array}{|c|c|c|c|c|c|c|}\hline a^2d+1 & a^2e+1 & abe+5 & ace\ \ldots & ade-5 & ae^2-1 & be^2-1 \\ abc-3 & abd+2 & acd-15 & ad^2-10 & bce+15 & bde-2 & cde+3 \\ b^3+2 & ac^2-9 & b^2d+10 & b^2e+10 & bd^2-10 & c^2e+9 & d^3-2 \\ & b^2c+6 & bc^2\ \ldots & bcd\ \ldots & c^2d\ \ldots & cd^2-6 & \\ & & & c^3\ \ldots & & & \\ \hline \pm 3 & \pm 9 & \pm 15 & \pm 10 & \pm 15 & \pm 9 & \pm 3\end{array}\;)\!(x, y)^6$$

No. 12.

$$\begin{array}{|ll|}\hline a^3e^3 & +1 \\ a^2bde^2 & -12 \\ a^2c^2e^2 & -18 \\ a^2cd^2e & +54 \\ a^2d^4 & -27 \\ ab^2ce^2 & +54 \\ ab^2d^2e & -6 \\ abc^2de & -180 \\ abcd^3 & +108 \\ ac^4e & +81 \\ ac^3d^2 & -54 \\ b^4e^2 & -27 \\ b^3cde & +108 \\ b^3d^3 & -64 \\ b^2c^3e & -54 \\ b^2c^2d^2 & +36 \\ bc^4d & \ldots \\ c^6 & \ldots \\ \hline \multicolumn{2}{c}{\pm 442}\end{array}$$

The tables Nos. 7, 8, 9, 10 and 11 are the irreducible covariants of a quartic. No. 7 is the quartic itself; No. 8 is the quadrinvariant; No. 9 is the quadricovariant, or Hessian; No. 10 is the cubinvariant; and No. 11 is the cubicovariant. The table No. 12 is the discriminant. And if we write No. $7 = U$, No. $8 = I$, No. $9 = H$, No. $10 = J$, No. $11 = \Phi$, No. $12 = \nabla$,

then the irreducible covariants are connected by the identical equation

$$JU^3 - IU^2H + 4H^3 + \Phi^2 = 0,$$

and we have

$$\nabla = I^3 - 27J^2.$$

[The Tables Nos. 13 to 24 which follow, and also Nos. 25 and 26 which are given in 143 relate to the binary quintic. I have inserted in the headings the capital letters A, B, ... L and also Q and Q′ by which I refer to these covariants of the quintic. A is the quintic itself, C is the Hessian, G is the quartinvariant, J a linear covariant: Q is the simplest octinvariant, and Q′ is the discriminant. As noticed in the original memoir we have $\mathrm{AI} + \mathrm{BF} - \mathrm{CE} = 0$; and $\mathrm{Q}' = \mathrm{G}^2 - 128\,\mathrm{Q}$, only the coefficient 128 was by mistake given as 1152.]

A. No. 13.

$a + 1$	$b + 5$	$c + 10$	$d + 10$	$e + 5$	$f + 1$

$)(x, y)^5$

B. No. 14.

$ae + 1$	$af + 1$	$bf + 1$
$bd - 4$	$be - 3$	$ce - 4$
$c^2 + 3$	$cd + 2$	$d^2 + 3$
± 4	± 3	± 4

$)(x, y)^2$

C. No. 15.

$ac + 1$	$ad + 3$	$ae + 3$	$af + 1$	$bf + 3$	$cf + 3$	$df + 1$
$b^2 - 1$	$bc - 3$	$bd + 3$	$be + 7$	$ce + 3$	$de - 3$	$e^2 - 1$
		$c^2 - 6$	$cd - 8$	$d^2 - 6$		
± 1	± 3	± 6	± 8	± 6	± 3	± 1

$)(x, y)^6$

D. No. 16.

abf ...	$acf + 1$	$adf + 1$	aef ...
$ace + 1$	$ade - 1$	$ae^2 - 1$	$bdf + 1$
$ad^2 - 1$	$b^2f - 1$	$bcf - 1$	$be^2 - 1$
$b^2e - 1$	$bce + 1$	$bde + 1$	$c^2f - 1$
$bcd + 2$	$bd^2 + 1$	$c^2e + 1$	$cde + 2$
$c^3 - 1$	$c^2d - 1$	$cd^2 - 1$	$d^3 - 1$
± 3	± 3	± 3	± 3

$)(x, y)^3$

E. No. 17.

$a^2f + 1$	$abf + 5$	$acf + 2$	$adf - 2$	$aef - 5$	$af^2 - 1$
$abe - 5$	$ace - 16$	$ade - 12$	$ae^2 - 8$	$bdf + 16$	$bef + 5$
$acd + 2$	$ad^2 + 6$	$b^2f + 8$	$bcf + 12$	$be^2 + 9$	$cdf - 2$
$b^2d + 8$	$b^2e - 9$	$bce - 38$	$bde + 38$	$c^2f - 6$	$ce^2 - 8$
$bc^2 - 6$	$bcd + 38$	$bd^2 + 72$	$c^2e - 72$	$cde - 38$	$d^2e + 6$
	$c^3 - 24$	$c^2d - 32$	$cd^2 + 32$	$d^3 + 24$	
± 11	± 49	± 82	± 82	± 49	± 11

$)(x, y)^5$

F. No. 18.

$a^2d + 1$	$a^2e + 2$	$a^2f + 1$	$abf + 7$	$acf + 5$	$adf - 5$	$aef - 7$	$af^2 - 1$	$bf^2 - 2$	$cf^2 - 1$
$abc - 3$	$abd + 1$	$abe + 11$	$ace - 8$	$ade - 40$	$ae^2 - 16$	$bdf + 8$	$bef - 11$	$cef - 1$	$def + 3$
$b^3 + 2$	$ac^2 - 12$	$acd - 34$	$ad^2 - 34$	$b^2f + 16$	$bcf + 40$	$be^2 - 29$	$cdf + 34$	$d^2f + 12$	$e^3 - 2$
	$b^2c + 9$	$b^2d + 16$	$b^2e + 29$	$bce + 47$	$bde - 47$	$c^2f + 34$	$ce^2 - 16$	$de^2 - 9$	
		$bc^2 + 6$	$bcd - 2$	$bd^2 - 44$	$c^2e + 44$	$cde + 2$	$d^2e - 6$		
			$c^3 + 8$	$c^2d + 16$	$cd^2 - 16$	$d^3 - 8$			
± 3	± 12	± 34	± 44	± 84	± 84	± 44	± 34	± 12	± 3

$)(x, y)^9$

G. No. 19.

$a^2f^2 + 1$
$abef - 10$
$acdf + 4$
$ace^2 + 16$
$ad^2e - 12$
$b^2df + 16$
$b^2e^2 + 9$
$bc^2f - 12$
$bcde - 76$
$bd^3 + 48$
$c^3e + 48$
$c^2d^2 - 32$
± 142

H. No. 20.

$a^2df + 1$	$a^2ef + 2$	$a^2f^2 + 1$	$abf^2 + 2$	$acf^2 + 1$
a^2e^2 ...	$abdf - 4$	$abef - 4$	$acef - 4$	$adef - 3$
$abcf - 3$	$abe^2 - 10$	$acdf - 2$	$ad^2f - 2$	$ae^3 + 2$
$abde - 5$	$ac^2f - 2$	$ace^2 + 4$	$ade^2 + 4$	b^2f^2 ...
$ac^2e + 10$	$acde + 24$	ad^2e ...	$b^2ef - 10$	$bcef - 5$
$acd^2 - 4$	$ad^3 - 12$	$b^2df + 4$	$bcdf + 24$	$bd^2f + 10$
$b^3f + 2$	$b^2cf + 4$	$b^2e^2 - 9$	$bce^2 + 16$	$bde^2 - 5$
$b^2ce - 5$	$b^2de + 16$	bc^2f ...	$bd^2e - 22$	$c^2df - 4$
$b^2d^2 + 14$	$bc^2e - 22$	$bcde + 50$	$c^3f - 12$	$c^2e^2 + 14$
$bc^2d - 16$	$bcd^2 - 4$	$bd^3 - 36$	$c^2de - 4$	$cd^2e - 16$
$c^4 + 6$	$c^3d + 8$	$c^3e - 36$	$cd^3 + 8$	$d^4 + 6$
		$c^2d^2 + 28$		
± 33	± 54	± 87	± 54	± 33

$)\!(x, y)^4$

I. No. 21.

$a^2cf + 1$	$a^2df + 2$	a^2ef ...	a^2f^2 ...	abf^2 ...	$acf^2 - 2$	$adf^2 - 1$
$a^2de - 1$	$a^2e^2 - 2$	$abdf + 2$	$abef$ ...	$acef - 2$	$adef$ ...	$ae^2f + 1$
$ab^2f - 1$	$abcf - 10$	$abe^2 - 2$	$acdf$ ...	$ad^2f + 1$	$ae^3 + 2$	$bcf^2 + 1$
$abce - 2$	$abde + 10$	$ac^2f - 1$	$ace^2 - 20$	$ade^2 + 1$	$b^2f^2 + 2$	$bdef + 2$
$abd^2 + 4$	ac^2e ...	$acde - 2$	$ad^2e + 20$	$b^2ef + 2$	$bcef$ ...	$be^3 - 3$
$ac^2d - 1$	acd^2 ...	$ad^3 + 3$	$b^2df + 20$	$bcdf + 2$	$bd^2f + 10$	$c^2ef - 4$
$b^3e + 3$	$b^3f - 2$	$b^2cf - 1$	b^2e^2 ...	$bce^2 - 5$	$bde^2 - 14$	$cd^2f + 1$
$b^2cd - 6$	$b^2ce + 14$	$b^2de + 5$	$bc^2f - 20$	$bd^2e - 1$	$c^2df - 10$	$cde^2 + 6$
$bc^3 + 3$	$b^2d^2 + 2$	$bc^2e + 1$	$bcde$ ...	$c^3f - 3$	$c^2e^2 - 2$	$d^3e - 3$
	$bc^2d - 26$	$bcd^2 - 9$	$bd^3 - 20$	$c^2de + 9$	$cd^2e + 26$	
	$c^4 + 12$	$c^3d + 4$	$c^3e + 20$	$cd^3 - 4$	$d^4 - 12$	
			c^2d^2 ...			
± 11	± 40	± 15	± 60	± 15	± 40	± 11

$)\!(x, y)^6.$

J. No. 22.

$a^2cf^2 + 1$	$a^2df^2 + 1$
$a^2def - 2$	$a^2e^2f - 1$
$a^2e^3 + 1$	$abcf^2 - 2$
$ab^2f^2 - 1$	$abdef - 4$
$abcef - 4$	$abe^3 + 6$
$abd^2f + 8$	$ac^2ef + 8$
$abde^2 - 2$	$acd^2f - 2$
$ac^2df - 2$	$acde^2 - 12$
$ac^2e^2 + 14$	$ad^3e + 6$
$acd^2e - 22$	$b^3f^2 + 1$
$ad^4 + 9$	$b^2cef - 2$
$b^3ef + 6$	$b^2d^2f + 14$
$b^2cdf - 12$	$b^2de^2 - 15$
$b^2ce^2 - 15$	$bc^2df - 22$
$b^2d^2e + 10$	$bc^2e^2 + 10$
$bc^3f + 6$	$bcd^2e + 30$
$bc^2de + 30$	$bd^4 - 15$
$bcd^3 - 20$	$c^4f + 9$
$c^4e - 15$	$c^3de - 20$
$c^3d^2 + 10$	$c^2d^3 + 10$
± 95	± 95

$)\!(x, y)^1$

K. No. 23.

a^2bf^2	...	a^2cf^2	+ 1	a^2df^2	− 1	a^2ef^2	...
a^2cef	+ 1	a^2def	− 5	a^2e^2f	+ 1	$abdf^2$	− 1
a^2d^2f	− 3	a^2e^3	+ 4	$abcf^2$	+ 5	abe^2f	+ 1
a^2de^2	+ 2	ab^2f^2	− 1	$abdef$	− 8	ac^2f^2	+ 3
ab^2ef	− 1	$abcef$	+ 8	abe^3	+ 3	$acdef$	− 14
$abcdf$	+ 14	abd^2f	+ 11	ac^2ef	− 11	ace^3	+ 8
$abce^2$	− 11	$abde^2$	− 17	acd^2f	+ 11	ad^3f	+ 9
abd^2e	− 1	ac^2df	− 11	$acde^2$	+ 6	ad^2e^2	− 6
ac^3f	− 9	ac^2e^2	− 16	ad^3e	− 6	b^2cf^2	− 2
ac^2de	+ 14	acd^2e	+ 44	b^3f^2	− 4	b^2def	+ 11
acd^3	− 6	ad^4	− 18	b^2cef	+ 17	b^2e^3	− 9
b^3df	− 8	b^3ef	− 3	b^2d^2f	+ 16	bc^2ef	+ 1
b^3e^2	+ 9	b^2cdf	− 6	b^2de^2	− 21	bcd^2f	− 14
b^2c^2f	+ 6	b^2ce^2	+ 21	bc^2df	− 44	$bcde^2$	+ 16
b^2cde	− 16	b^2d^2e	− 5	bc^2e^2	+ 5	bd^3e	− 3
b^2d^3	+ 8	bc^3f	+ 6	bcd^2e	+ 39	c^3df	+ 6
bc^3e	+ 3	bc^2de	− 39	bd^4	− 12	c^3e^2	− 8
bc^2d^2	− 2	bcd^3	+ 22	c^4f	+ 18	c^2d^2e	+ 2
c^4d	...	c^4e	+ 12	c^3de	− 22	cd^4	...
		c^3d^2	− 8	c^2d^3	+ 8		
± 57		± 129		± 129		± 57	

$(\ldots)(x, y)^3$

L. No. 24.

a^3ef	...	a^3f^2	...	a^2bf^2	...	a^2cf^2	− 1	a^2df^2	+ 1	a^2ef^2	...	a^2f^3	...	abf^3	...
a^2bdf	...	a^2bef	...	a^2cef	− 3	a^2def	+ 7	a^2e^2f	− 1	$abdf^2$	+ 3	$abef^2$	...	$acef^2$	...
a^2be^2	...	a^2cdf	+ 7	a^2d^2f	+ 12	a^2e^3	− 6	$abcf^2$	− 7	abe^2f	− 3	$acdf^2$	− 7	ad^2f^2	− 2
a^2c^2f	+ 2	a^2ce^2	− 10	a^2de^2	− 9	ab^2f^2	+ 1	$abdef$	+ 26	ac^2f^2	− 12	ace^2f	+ 7	ade^2f	+ 4
a^2cde	− 5	a^2d^2e	+ 3	ab^2ef	+ 3	$abcef$	− 26	abe^3	− 19	$acdef$	+ 18	ad^2ef	+ 7	ae^4	− 2
a^2d^3	+ 3	ab^2df	− 7	$abcdf$	− 18	abd^2f	+ 32	ac^2ef	− 32	ace^3	+ 6	ade^3	− 7	b^2ef^2	...
ab^2cf	− 4	ab^2e^2	+ 10	$abce^2$	− 18	$abde^2$	− 8	acd^2f	+ 18	ad^3f	+ 3	b^2df^2	+ 10	$bcdf^2$	+ 5
ab^2de	+ 5	abc^2f	− 7	abd^2e	+ 30	ac^2df	− 18	$acde^2$	+ 53	ad^2e^2	− 15	b^2e^2f	− 10	bce^2f	− 5
abc^2e	+ 5	$abcde$	− 8	ac^3f	− 3	ac^2e^2	+ 6	ad^3e	− 39	b^2cf^2	+ 9	bc^2f^2	− 3	bd^2ef	− 5
$abcd^2$	− 7	abd^3	+ 9	ac^2de	+ 45	acd^2e	+ 52	b^3f^2	+ 6	b^2def	+ 18	$bcdef$	+ 8	bde^3	+ 5
ac^3d	+ 1	ac^3e	+ 22	acd^3	− 39	ad^4	− 39	b^2cef	+ 8	b^2e^3	− 27	bce^3	− 2	c^3f^2	− 3
b^4f	+ 2	ac^2d^2	− 19	b^3df	− 6	b^3ef	+ 19	b^2d^2f	− 6	bc^2ef	− 30	bd^3f	− 22	c^2def	+ 7
b^3ce	− 5	b^3cf	+ 7	b^3e^2	+ 27	b^2cdf	− 53	b^2de^2	− 20	bcd^2f	− 45	bd^2e^2	+ 19	c^2e^3	+ 2
b^3d^2	− 2	b^3de	+ 2	b^2c^2f	+ 15	b^2ce^2	+ 20	bc^2df	+ 45	$bcde^2$	+ 87	c^3ef	− 9	cd^3f	− 1
b^2c^2d	+ 8	b^2c^2e	− 19	b^2cde	− 87	b^2d^2e	− 25	bc^2e^2	+ 25	bd^3e	− 12	c^2d^2f	+ 19	cd^2e^2	− 8
bc^4	− 3	b^2cd^2	− 11	b^2d^3	+ 6	bc^3f	+ 39	bcd^2e	− 52	c^3df	+ 39	c^2de^2	+ 11	d^4e	+ 3
		bc^3d	+ 33	bc^3e	+ 12	bc^2de	− 45	bd^4	...	c^3e^2	− 6	cd^3e	− 33		
		c^5	− 12	bc^2d^2	+ 57	bcd^3	+ 65	c^4f	+ 39	c^2d^2e	− 57	d^5	+ 12		
				c^4d	− 24	c^4e	...	c^3de	− 65	cd^4	+ 24				
						c^3d^2	− 20	c^2d^3	+ 20						
± 26		± 93		± 207		± 241		± 241		± 207		± 93		± 26	

$(\ldots)(x, y)^7$

No. 25, $Q = +1\, a^3cdf^3 + \&c.$
No. 26, $Q' = +1\, a^4f^4 \quad + \&c.$ } [see *post*, 143.]