156.

A FIFTH MEMOIR UPON QUANTICS.

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THE present memoir was originally intended to contain a development of the theories of the covariants of certain binary quantics, viz. the quadric, the cubic, and the quartic; but as regards the theories of the cubic and the quartic, it was found necessary to consider the case of two or more quadrics, and I have therefore comprised such systems of two or more quadrics, and the resulting theories of the harmonic relation and of involution, in the subject of the memoir; and although the theory of homography or of the anharmonic relation belongs rather to the subject of bipartite binary quadrics, yet from its connexion with the theories just referred to, it is also considered in the memoir. The paragraphs are numbered continuously with those of my former memoirs on the subject: Nos. 92 to 95 relate to a single quadric; Nos. 96 to 114 to two or more quadrics, and the theories above referred to; Nos. 115 to 127 to the cubic, and Nos. 128 to 145 to the quartic. The several quantics are considered as expressed not only in terms of the coefficients, but also in terms of the roots;—and I consider the question of the determination of their linear factors,—a question, in effect, identical with that of the solution of a quadric, cubic, or biquadratic equation. The expression for the linear factor of a quadric is deduced from a well-known formula; those for the linear factors of a cubic and a quartic were first given in my "Note sur les Covariants d'une fonction quadratique, cubique ou biquadratique à deux indéterminées," Crelle, vol. L. (1855), pp. 285—287, [135]. It is remarkable that they are in one point of view more simple than the expression for the linear factor of a quadric.

92. In the case of a quadric the expressions considered are

$$(a,\ b,\ c \between x,\ y)^2, \qquad (1)$$

$$ac-b^2\ , \qquad (2)$$

where (1) is the quadric, and (2) is the discriminant, which is also the quadrinvariant, catalecticant, and Hessian.

And where it is convenient to do so, I write

$$(1) = U,$$

$$(2) = \square.$$

93. We have

$$(\partial_c,\ -\partial_b,\ \partial_a \between x,\ y)^2 \square = U,$$

which expresses that the evectant of the discriminant is equal to the quadric;

$$(a,\ b,\ c \between \partial_y,\ -\partial_x)^2\, U = 4\square,$$

which expresses that the provectant of the quadric is equal to the discriminant;

$$(a,\ b,\ c \between bx + cy,\ -ax - by)^2 = \square\, U,$$

which expresses that a transmutant of the quadric is equal to the product of the quadric and the discriminant.

94. When the quadric is expressed in terms of the roots, we have

$$a^{-1}U = (x - \alpha y)(x - \beta y),$$

$$a^{-2}\square = -\tfrac{1}{4}(\alpha - \beta)^2;$$

and in the case of a pair of equal roots,

$$a^{-1}U = (x - \alpha y)^2,$$

$$\square \quad = 0.$$

95. The problem of the solution of a quadratic equation is that of finding a linear factor of the quadric. To obtain such linear factor in a symmetrical form, it is necessary to introduce arbitrary quantities which do not really enter into the solution, and the form obtained is thus in some sort more complicated than in the like problem for a cubic or a quartic. The solution depends on the linear transformation of the quadric, viz. if we write

$$(a,\ b,\ c \between \lambda x + \mu y,\ \nu x + \rho y)^2 = (a',\ b',\ c' \between x,\ y)^2,$$

so that

$$a' = (a,\ b,\ c \between \lambda,\ \nu)^2,$$

$$b' = (a,\ b,\ c \between \lambda,\ \nu \between \mu,\ \rho),$$

$$c' = (a,\ b,\ c \between \mu,\ \rho)^2,$$

then

$$a'c' - b'^2 = (ac - b^2)(\lambda\rho - \mu\nu)^2,$$

an equation which in a different notation is

$$(a,\ b,\ c \between x,\ y)^2 . (a,\ b,\ c \between X,\ Y)^2 - \{(a,\ b,\ c \between x,\ y \between X,\ Y)\}^2 = \square\, (Yx - Xy)^2,$$

in which form it is a theorem relating to the quadric and its first and second emanants. The equation shows that

$$(a,\ b,\ c \between x,\ y \between X,\ Y) + \sqrt{-\square}\,(Yx - Xy),$$

where $(X,\ Y)$ are treated as supernumerary arbitrary constants, is a linear factor of $(a,\ b,\ c \between x,\ y)^2$, and this is the required solution.

96. In the case of two quadrics, the expressions considered are

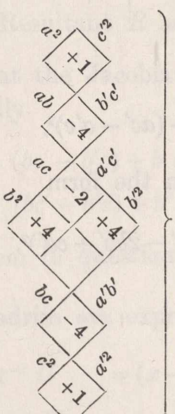
$$(a\ ,\ b\ ,\ c \between x,\ y)^2, \qquad (1)$$

$$(a',\ b',\ c' \between x,\ y)^2, \qquad (2)$$

$$a\,c - b^2\ , \qquad (3)$$

$$a\,c' - 2bb' + ca', \qquad (4)$$

$$a'c' - b'^2\ , \qquad (5)$$



, (6)

$$(ab' - a'b,\quad ac' - a'c\ ,\ bc' - b'c \between x,\ y)^2, \qquad (7)$$

$$(\lambda a + \mu a',\quad \lambda b + \mu b'\ ,\ \lambda c + \mu c' \between x,\ y)^2, \qquad (8)$$

$$(ac - b^2\ ,\ ac' - 2bb' + ca',\ a'c' - b'^2 \between \lambda,\ \mu)^2, \qquad (9)$$

(1) and (2) are the quadrics, (3) and (5) are the discriminants, and (4) is the lineo-linear invariant, or connective of the discriminants; (6) is the resultant of the two quadrics, (7) is the Jacobian, (8) is an intermediate, and (9) is the discriminant of the intermediate. And where it is convenient to do so, I write

$$\begin{aligned}(1) &= U,\\ (2) &= U',\\ (3) &= \square,\\ (4) &= Q,\\ (5) &= \square',\\ (6) &= R,\\ (7) &= H,\\ (8) &= W,\\ (9) &= \Theta.\end{aligned}$$

97. The Jacobian (7) may also be written in the form

$$\begin{vmatrix} y^2, & -yx, & x^2 \\ a\ , & b\ , & c \\ a', & b', & c' \end{vmatrix}.$$

The Resultant (6) may be written in the form

$$\begin{vmatrix} & a\ , & 2b\ , & c \\ a\ , & 2b\ , & c\ , & \\ & a', & 2b', & c' \\ a', & 2b', & c', & \end{vmatrix},$$

and also, taken negatively, in the form

$$4(ab' - a'b)(bc' - b'c) - (ac' - a'c)^2,$$

which is the discriminant of the Jacobian; and in the form

$$4(ac - b^2)(a'c' - b'^2) - (ac' - 2bb' + ca')^2,$$

which is the discriminant of the Intermediate.

98. We have the following relations:

$$\begin{aligned} (a,\ b,\ c \between b'x + c'y,\ -a'x - b'y)^2 &= -(a'c' - b'^2) \quad (a\ ,\ b\ ,\ c \between x,\ y)^2 \\ &\quad + (ac' - 2bb' + ca')\ (a',\ b',\ c' \between x,\ y)^2, \\ (a',\ b',\ c' \between bx + cy,\ -ax - by)^2 &= +(ac' - 2bb' + ca')\ (a\ ,\ b\ ,\ c \between x,\ y)^2 \\ &\quad - (ac - b^2) \quad (a',\ b',\ c' \between x,\ y)^2, \end{aligned}$$

and moreover

$$\begin{aligned} &(ac - b^2,\ ac' - 2bb' + ca',\ a'c' - b'^2 \between U',\ -U)^2 \\ &\qquad = -\{(ab' - a'b,\ ac' - a'c,\ bc' - b'c \between x,\ y)^2\}^2, \end{aligned}$$

an equation, the interpretation of which will be considered in the sequel.

99. The most important relations which may exist between the two quadrics are:

First, when the connective vanishes, or

$$ac' - 2bb' + ca' = 0,$$

in which case the two quadrics are said to be *harmonically* related: the nature of this relation will be further considered.

Secondly, when $R=0$, the two quadrics have in this case a common root, which is given by any of the equations,

$$\begin{aligned} x^2 : 2xy : y^2 &= \partial_a R \quad : \partial_b R \quad : \partial_c R \\ &= \partial_{a'} R \quad : \partial_{b'} R \quad : \partial_{c'} R \\ &= bc' - b'c : ca' - c'a : ab' - a'b. \end{aligned}$$

The last set of values express that the Jacobian is a perfect square, and that the two roots are each equal to the common root of the two quadrics.

The preceding values of the ratios $x^2 : 2xy : y^2$ are consistent with each other in virtue of the assumed relation $R=0$, hence in general the functions

$$4\partial_a R \,.\, \partial_c R - (\partial_b R)^2,\ \partial_a R \,.\, \partial_{b'} R - \partial_b R \,.\, \partial_{a'} R, \text{ \&c.}$$

all of them contain the Resultant R as a factor.

It is easy to see that the Jacobian is harmonically related to each of the quadrics; in fact we have identically

$$\begin{aligned} a\,(bc' - b'c) + b\,(ca' - c'a) + c\,(ab' - a'b) &= 0, \\ a'\,(bc' - b'c) + b'\,(ca' - c'a) + c'\,(ab' - a'b) &= 0, \end{aligned}$$

which contain the theorem in question.

100. When the quadrics are expressed in terms of the roots, we have

$$\begin{aligned} a^{-1}\,U \quad &= (x-\alpha y)(x-\beta y), \\ a'^{-1}\,U' \quad &= (x-\alpha' y)(x-\beta' y), \\ 4\,a^{-2}\,\square \quad &= -(\alpha-\beta)^2, \\ 2\,(aa')^{-1} Q &= 2\alpha\beta + 2\alpha'\beta' - (\alpha+\beta)(\alpha'+\beta'), \\ 4\,a'^{-2}\,\square' \quad &= -(\alpha'-\beta')^2, \\ (aa')^{-2}\,R &= (\alpha-\alpha')(\alpha-\beta')(\beta-\alpha')(\beta-\beta'), \\ (aa')^{-1}\,H &= \begin{vmatrix} y^2, & 2yx\,, & x^2 \\ 1\,, & \alpha+\beta, & \alpha\beta \\ 1\,, & \alpha'+\beta', & \alpha'\beta' \end{vmatrix} \end{aligned}$$

101. The comparison of the last-mentioned value of R with the expression in terms of the roots obtained from the equation

$$-R = 4\square\,\square' - Q^2,$$

gives the identical equation

$$(\alpha-\beta)^2(\alpha'-\beta')^2 - \{2\alpha\beta + 2\alpha'\beta' - (\alpha+\beta)(\alpha'+\beta')\}^2 = -4\,(\alpha-\alpha')(\alpha-\beta')(\beta-\alpha')(\beta-\beta'),$$

which may be easily verified.

102. We have identically

$$\begin{aligned} &2\alpha\beta + 2\alpha'\beta' - (\alpha+\beta)(\alpha'+\beta') \\ &= 2(\alpha-\alpha')(\alpha-\beta') - (\alpha-\beta)(2\alpha-\alpha'-\beta') \\ &= 2(\beta-\alpha')(\beta-\beta') - (\beta-\alpha)(2\beta-\alpha'-\beta') \\ &= 2(\alpha'-\alpha)(\alpha'-\beta) - (\alpha'-\beta')(2\alpha'-\alpha-\beta) \\ &= 2(\beta'-\alpha)(\beta'-\beta) - (\beta'-\alpha')(2\beta'-\alpha-\beta); \end{aligned}$$

and the equation $Q = ac' - 2bb' + ca' = 0$ may consequently be written in the several forms

$$\frac{2}{\alpha-\beta} = \frac{1}{\alpha-\alpha'} + \frac{1}{\alpha-\beta},$$

$$\frac{2}{\beta-\alpha} = \frac{1}{\beta-\alpha'} + \frac{1}{\beta'-\beta},$$

$$\frac{2}{\alpha'-\beta'} = \frac{1}{\alpha'-\alpha} + \frac{1}{\alpha'-\beta},$$

$$\frac{2}{\beta'-\alpha'} = \frac{1}{\beta'-\alpha} + \frac{1}{\beta'-\beta},$$

so that the roots (α, β), (α', β') are harmonically related to each other, and hence the notion of the harmonic relation of the two quadrics.

103. In the case where the two quadrics have a common root $\alpha = \alpha'$,

$$\begin{aligned} a^{-1}\,U &= (x-\alpha y)(x-\beta y), \\ a'^{-1}\,U' &= (x-\alpha y)(x-\beta' y), \\ 4\,a^{-2}\,\square &= -(\alpha-\beta)^2, \\ 2\,(aa')^{-1}Q &= (\alpha-\beta)(\alpha-\beta'), \\ 4a'^{-2}\,\square' &= -(\alpha-\beta')^2, \\ R &= 0, \\ (aa')^{-1}H &= (\beta'-\beta)(x-\alpha y)^2. \end{aligned}$$

104. In the case of three quadrics, of the expressions which are or might be considered, it will be sufficient to mention

$$(a\,,\ b\,,\ c\ \between x,\ y)^2, \tag{1}$$

$$(a',\ b',\ c'\ \between x,\ y)^2, \tag{2}$$

$$(a'',\ b'',\ c''\between x,\ y)^2, \tag{3}$$

$$\begin{vmatrix} a\,, & b\,, & c \\ a', & b', & c' \\ a'', & b'', & c'' \end{vmatrix}, \tag{4}$$

where (1), (2), (3) are the quadrics themselves, and (4) is an invariant, linear in the coefficients of each quadric. And where it is convenient to do so, I write

$$(1) = U,$$
$$(2) = U',$$
$$(3) = U'',$$
$$(4) = \Omega.$$

105. The equation $\Omega = 0$ is, it is clear, the condition to be satisfied by the coefficients of the three quadrics, in order that there may be a syzygetic relation $\lambda U + \mu U' + \nu U'' = 0$, or what is the same thing, in order that each quadric may be an intermediate of the other two quadrics; or again, in order that the three quadrics may be *in Involution.* Expressed in terms of the roots, the relation is

$$\begin{vmatrix} 1, & \alpha + \beta, & \alpha\beta \\ 1, & \alpha' + \beta', & \alpha'\beta' \\ 1, & \alpha'' + \beta'', & \alpha''\beta'' \end{vmatrix} = 0;$$

and when this equation is satisfied, the three pairs, or as it is usually expressed, the six quantities α, β; α', β'; α'', β'', are said to be in involution, or to form an involution. And the two perfectly arbitrary pairs α, β; α', β' considered as belonging to such a system, may be spoken of as an involution. If the two terms of a pair are equal, e.g. if $\alpha'' = \beta'' = \theta$, then the relation is

$$\begin{vmatrix} 1, & 2\theta, & \theta^2 \\ 1, & \alpha + \beta, & \alpha\beta \\ 1, & \alpha' + \beta', & \alpha'\beta' \end{vmatrix} = 0;$$

and such a system is sometimes spoken of as an involution of five terms. Considering the pairs (α, β), (α', β') as given, there are of course two values of θ which satisfy the preceding equation; and calling these $\theta_{,}$ and $\theta_{,,}$, then $\theta_{,}$ and $\theta_{,,}$ are said to be the sibiconjugates of the involution α, β; α', β'. It is easy to see that $\theta_{,}$, $\theta_{,,}$ are the roots of the equation $H = 0$, where H is the Jacobian of the two quadrics U and U' whose roots are (α, β), (α', β'). In fact, the quadric whose roots are $\theta_{,}$, $\theta_{,,}$ is

$$\begin{vmatrix} y^2, & 2yx, & x^2 \\ 1, & \alpha + \beta, & \alpha\beta \\ 1, & \alpha' + \beta', & \alpha'\beta' \end{vmatrix}$$

which has been shown to be the Jacobian in question. But this may be made clearer as follows:—If we imagine that λ, μ are determined in such manner that the intermediate $\lambda U + \mu U'$ may be a perfect square, then we shall have $\lambda U + \mu U' = a'' (x - \theta y)^2$, where θ denotes one or other of the sibiconjugates $\theta_{,}$, $\theta_{,,}$ of the involution. But the condition in order that $\lambda U + \mu U'$ may be a square is

$$(ac - b^2, \; ac' - 2bb' + ca', \; a'c' - b'^2 \between \lambda, \mu)^2;$$

and observing that the equation $\lambda : \mu = U' : -U$ implies $\lambda U + \mu U' = 0 = a''(x-\theta y)^2$, it is obvious that the function

$$(ac-b^2,\ ac'-2bb'+ca',\ a'c'-b'^2 \between U',\ -U)^2$$

must be to a factor *près* equal to $(x-\theta_{,}y)^2(x-\theta_{,,}y)^2$. But we have identically

$$(ac-b^2,\ ac'-2bb'+ca',\ a'c'-b'^2 \between U',\ -U)^2 = -\{(ab'-a'b,\ ac'-a'c,\ bc'-b'c \between x,\ y)^2\}^2,$$

and we thus see that $(x-\theta_{,}y)$, $(x-\theta_{,,}y)$ are the factors of the Jacobian.

106. It has been already remarked that the Jacobian is harmonically related to each of the quadrics U, U'; hence we see that the sibiconjugates $\theta_{,}$, $\theta_{,,}$ of the involution α, β, α', β' are a pair harmonically related to the pair α, β, and also harmonically related to the pair α', β', and this properly might be taken as the definition for the sibiconjugates $\theta_{,}$, $\theta_{,,}$ of an involution of four terms. And moreover, α, β; α', β' being given, and $\theta_{,}$, $\theta_{,,}$ being determined as the sibiconjugates of the involution, if α'', β'' be a pair harmonically related to $\theta_{,}$, $\theta_{,,}$, then the three pairs α, β; α', β'; α'', β'' will form an involution; or what is the same thing, any three pairs α, β; α', β'; α'', β'', each of them harmonically related to a pair $\theta_{,}$, $\theta_{,,}$, will be an involution, and $\theta_{,}$, $\theta_{,,}$ will be the sibiconjugates of the involution.

107. In particular, if α, β be harmonically related to $\theta_{,}$, $\theta_{,,}$, then it is easy to see that $\theta_{,}$, $\theta_{,}$ may be considered as harmonically related to $\theta_{,}$, $\theta_{,,}$, and in like manner $\theta_{,,}$, $\theta_{,,}$ will be harmonically related to $\theta_{,}$, $\theta_{,,}$; that is, the pairs $\theta_{,}$, $\theta_{,}$; $\theta_{,,}$, $\theta_{,,}$ and α, β will form an involution. This comes to saying that the equation

$$\begin{vmatrix} 1, & 2\theta_{,}\,, & \theta_{,}^2 \\ 1, & 2\theta_{,,}\,, & \theta_{,,}^2 \\ 1, & \alpha+\beta, & \alpha\beta \end{vmatrix} = 0$$

is equivalent to the harmonic relation of the pairs α, β; $\theta_{,}$, $\theta_{,,}$; and in fact the determinant is

$$(\theta_{,}-\theta_{,,})\left(2\alpha\beta + 2\theta_{,}\theta_{,,} - (\alpha+\beta)(\theta_{,}+\theta_{,,})\right),$$

which proves the theorem in question.

108. Before proceeding further, it is proper to consider the equation

$$\begin{vmatrix} 1, & \alpha, & \alpha', & \alpha\alpha' \\ 1, & \beta, & \beta', & \beta\beta' \\ 1, & \gamma, & \gamma', & \gamma\gamma' \\ 1, & \delta, & \delta', & \delta\delta' \end{vmatrix} = 0,$$

which expresses that the sets $(\alpha, \beta, \gamma, \delta)$ and $(\alpha', \beta', \gamma', \delta')$ are homographic; for although the homographic equation may be considered as belonging to the theory of the bipartite quadric $(x-\alpha y)(\mathrm{x}-\alpha'\mathrm{y})$, yet the theory of involution cannot be completely discussed except in connexion with that of homography. If we write

$$A=(\beta-\gamma)(\alpha-\delta),\quad B=(\gamma-\alpha)(\beta-\delta),\quad C=(\alpha-\beta)(\gamma-\delta),$$
$$A'=(\beta'-\gamma')(\alpha'-\delta'),\quad B'=(\gamma'-\alpha')(\beta'-\delta'),\quad C'=(\alpha'-\beta')(\gamma'-\delta'),$$

then we have

$$A+B+C=0,$$
$$A'+B'+C'=0,$$

and thence

$$BC'-B'C=CA'-C'A=AB'-A'B;$$

and either of these expressions is in fact equal to the last-mentioned determinant, as may be easily verified. Hence, when the determinant vanishes, we have

$$A : B : C = A' : B' : C'.$$

Any one of the three ratios $A : B : C$, for instance the ratio $B : C$, $=$

$$\frac{(\gamma-\alpha)(\beta-\delta)}{(\alpha-\beta)(\gamma-\delta)},$$

is said to be the anharmonic ratio of the set $(\alpha, \beta, \gamma, \delta)$, and consequently the two sets $(\alpha, \beta, \gamma, \delta)$ and $(\alpha', \beta', \gamma', \delta')$ will be homographically related when the anharmonic ratios (that is, the corresponding anharmonic ratios) of the two sets are equal.

If any one of the anharmonic ratios be equal to unity, then the four terms of the set taken in a proper manner in pairs, will be harmonics; thus the equation $\frac{B}{C}=1$ gives

$$\frac{(\gamma-\alpha)(\beta-\delta)}{(\alpha-\beta)(\gamma-\delta)}=1,$$

which is reducible to

$$2\alpha\delta+2\beta\gamma-(\alpha+\delta)(\beta+\gamma)=0,$$

which expresses that the pairs α, δ and β, γ are harmonics.

109. Now returning to the theory of involution (and for greater convenience taking α, α' &c. instead of α, β &c. to represent the terms of the same pair), the pairs α, α'; β, β'; γ, γ'; δ, δ'; &c. will be in involution if each of the determinants formed with any three lines of the matrix

$$\begin{array}{lll} 1, & \alpha+\alpha', & \alpha\alpha', \\ 1, & \beta+\beta', & \beta\beta', \\ 1, & \gamma+\gamma', & \gamma\gamma', \\ 1, & \delta+\delta', & \delta\delta', \\ \&c. & & \end{array}$$

vanishes: but this being so, the determinant

$$\begin{vmatrix} 1, & \alpha, & \alpha', & \alpha\alpha' \\ 1, & \beta, & \beta', & \beta\beta' \\ 1, & \gamma, & \gamma', & \gamma\gamma' \\ 1, & \delta, & \delta', & \delta\delta' \end{vmatrix}$$

which is equal to

$$\begin{vmatrix} \alpha, & 1, & \alpha+\alpha', & \alpha\alpha' \\ \beta, & 1, & \beta+\beta', & \beta\beta' \\ \gamma, & 1, & \gamma+\gamma', & \gamma\gamma' \\ \delta, & 1, & \delta+\delta', & \delta\delta' \end{vmatrix}$$

will vanish, or the two sets $(\alpha, \beta, \gamma, \delta)$ and $(\alpha', \beta', \gamma', \delta')$ will be homographic; that is, if any number of pairs are in involution, then, considering four pairs and selecting in any manner a term out of each pair, these four terms and the other terms of the same four pairs form respectively two sets, and the two sets so obtained will be homographic.

110. In particular, if we have only three pairs α, α'; β, β'; γ, γ', then the sets $\alpha, \beta, \gamma, \alpha'$ and $\alpha', \beta', \gamma', \alpha$ will be homographic; in fact, the condition of homography is

$$\begin{vmatrix} 1, & \alpha, & \alpha', & \alpha\alpha' \\ 1, & \beta, & \beta', & \beta\beta' \\ 1, & \gamma, & \gamma', & \gamma\gamma' \\ 1, & \alpha', & \alpha, & \alpha\alpha' \end{vmatrix} = 0,$$

which may be written

$$\begin{vmatrix} \alpha, & 1, & \alpha+\alpha', & \alpha\alpha' \\ \beta, & 1, & \beta+\beta', & \beta\beta' \\ \gamma, & 1, & \gamma+\gamma', & \gamma\gamma' \\ \alpha', & 1, & \alpha+\alpha', & \alpha\alpha' \end{vmatrix} = 0,$$

or what is the same thing,

$$\begin{vmatrix} \alpha & , & 1, & \alpha+\alpha', & \alpha\alpha' \\ \beta & , & 1, & \beta+\beta', & \beta\beta' \\ \gamma & , & 1, & \gamma+\gamma', & \gamma\gamma' \\ \alpha'-\alpha, & & 0, & 0 & , 0 \end{vmatrix} = 0,$$

so that the first-mentioned relation is equivalent to

$$(\alpha'-\alpha) \begin{vmatrix} 1, & \alpha+\alpha', & \alpha\alpha' \\ 1, & \beta+\beta', & \beta\beta' \\ 1, & \gamma+\gamma', & \gamma\gamma' \end{vmatrix} = 0,$$

and the two sets give rise to an involution. The condition of homography as expressed by the equality of the anharmonic ratios may be written

$$\frac{\alpha-\beta\,.\,\gamma-\alpha'}{\alpha-\gamma\,.\,\alpha'-\beta}=\frac{\alpha'-\beta'\,.\,\gamma'-\alpha}{\alpha'-\gamma'\,.\,\alpha-\beta'};$$

or multiplying out,

$$(\alpha-\beta)(\alpha-\beta')(\alpha'-\gamma)(\alpha'-\gamma')-(\alpha'-\beta)(\alpha'-\beta')(\alpha-\gamma)(\alpha-\gamma')=0,$$

which is a form for the equation of involution of the three pairs. But this and the other transformations of the equation of involution is best obtained by a different method, as will be presently seen.

111. Imagine now any number of pairs α, α'; β, β'; γ, γ'; δ, δ'; &c. in involution, and let x, y, z, w be the fourth harmonics of the same quantity λ with respect to the pairs α, α'; β, β'; γ, γ' and δ, δ' respectively; then the anharmonic ratios of the set (x, y, z, w) will be independent of λ, or what is the same thing, if x', y', z', w' are the fourth harmonics of any other quantity λ' with respect to the same four pairs, the sets (x, y, z, w) and (x', y', z', w') will be homographic, or we shall have

$$\begin{vmatrix} 1, & x, & x', & xx' \\ 1, & y, & y', & yy' \\ 1, & z, & z', & zz' \\ 1, & w, & w', & ww' \end{vmatrix}=0.$$

It will be sufficient to show this in the case where λ is anything whatever, but λ' has a determinate value, say $\lambda'=\infty$; and since if all the terms α, α', &c. are diminished by the same quantity λ the relations of involution and homography will not be affected, we may without loss of generality assume $\lambda=0$, but in this case

$$x=\frac{2\alpha\alpha'}{\alpha+\alpha'},\quad x'=\tfrac{1}{2}(\alpha+\alpha'),$$

and the equation to be proved is

$$\begin{vmatrix} 1, & \dfrac{\alpha\alpha'}{\alpha+\alpha'}, & \alpha+\alpha', & \alpha\alpha' \\ 1, & \dfrac{\beta\beta'}{\beta+\beta'}, & \beta+\beta', & \beta\beta' \\ 1, & \dfrac{\gamma\gamma'}{\gamma+\gamma'}, & \gamma+\gamma', & \gamma\gamma' \\ 1, & \dfrac{\delta\delta'}{\delta+\delta'}, & \delta+\delta', & \delta\delta' \end{vmatrix}=0,$$

which is obviously a consequence of the equations which express the involution of the four pairs.

A set homographic with $x, y, z\ w$, which are the fourth harmonics of any quantity whatever λ with respect to the pairs in involution, α, α'; β, β'; γ, γ'; δ, δ', is said to be homographic with the four pairs, and we have thus the notion of a set of single quantities homographic with a set of pairs in involution. This very important theory is due to M. Chasles.

112. Let r; s; t be the anharmonic ratios of a set $\alpha, \beta, \gamma, \delta$, and let $r_,$; $s_,$; $t_,$ be the anharmonic ratios (corresponding or not corresponding) of a set $\alpha_,, \beta_,, \gamma_,, \delta_,$. And suppose that r'; s'; t'; $r'_,$; $s'_,$; $t'_,$; r''; s''; t''; $r''_,$; $s''_,$; $t''_,$; r'''; s'''; t'''; $r'''_,$; $s'''_,$; $t'''_,$, are the analogous quantities for three other pairs of sets; then an equation such as

$$\begin{vmatrix} 1, & \dfrac{r}{s}, & \dfrac{r_,}{s_,}, & \dfrac{rr_,}{ss_,} \\ \vdots & & & \end{vmatrix} = 0,$$

or as it is more conveniently written,

$$\begin{vmatrix} ss_,\ , & rs_,\ , & r_,s\ , & rr_, \\ s's'_,\ , & r's'_,\ , & r'_,s'\ , & r'r'_, \\ s''s''_,\ , & r''s''_,\ , & r''_,s''\ , & r''r''_, \\ s'''s'''_,\ , & r'''s'''_,\ , & r'''_,s'''\ , & r'''r'''_, \end{vmatrix} = 0$$

is a relation independent of the particular ratios $r : s$ which have been chosen for the anharmonic ratios of the sets; this is easily shown by means of the equations

$$r + s + t = 0, \quad r_, + s_, + t_, = 0,$$

which connect the anharmonic ratios. The equation in fact expresses a certain relation between four sets $(\alpha, \beta, \gamma, \delta)$ and four other sets $(\alpha_,, \beta_,, \gamma_,, \delta_,)$; a relation which may be termed the relation of the homography of the anharmonic ratios of four and four sets: the notion of this relation is also due to M. Chasles.

113. The general relation

$$\begin{vmatrix} 1, & \alpha + \beta\ , & \alpha\beta \\ 1, & \alpha' + \beta', & \alpha'\beta' \\ 1, & \alpha'' + \beta'', & \alpha''\beta'' \end{vmatrix} = 0$$

may be exhibited in a great variety of forms. In fact, if the determinant is denoted by Υ, then multiplying by this determinant the two sides of the identical equation

$$\begin{vmatrix} u^2, & -u, & 1 \\ v^2, & -v\ , & 1 \\ w^2, & -w, & 1 \end{vmatrix} = (u - v)(v - w)(w - u),$$

we obtain

$$\Upsilon\,(u - v)(v - w)(w - u) = \begin{vmatrix} (u - \alpha\ )(u - \beta\ ), & (v - \alpha\ )(v - \beta\ ), & (w - \alpha\ )(w - \beta\ ) \\ (u - \alpha'\ )(u - \beta'), & (v - \alpha'\ )(v - \beta'), & (w - \alpha'\ )(w - \beta') \\ (u - \alpha'')(u - \beta''), & (v - \alpha'')(v - \beta''), & (w - \alpha'')(w - \beta'') \end{vmatrix}.$$

If, for example, $u=\alpha$, $v=\beta$, then we have

$$\Upsilon(\alpha-\beta)=-(\alpha-\alpha')(\alpha-\beta')(\beta-\alpha'')(\beta-\beta'')+(\beta-\alpha')(\beta-\beta')(\alpha-\alpha'')(\alpha-\beta'');$$

and again, if $u=\alpha$, $v=\alpha'$, $w=\alpha''$, then we have

$$\Upsilon=-(\alpha-\beta'')(\alpha'-\beta)(\alpha''-\beta')+(\alpha-\beta')(\alpha'-\beta'')(\alpha''-\beta).$$

Putting $\Upsilon=0$, the two equations give respectively

$$\frac{(\alpha-\alpha')(\beta-\alpha'')}{(\alpha-\alpha'')(\alpha'-\beta)}=\frac{(\alpha-\beta'')(\beta-\beta')}{(\alpha-\beta')(\beta''-\beta)};$$

and

$$(\alpha-\beta'')(\alpha'-\beta)(\alpha''-\beta')=(\alpha-\beta')(\alpha'-\beta'')(\alpha''-\beta),$$

which are both of them well-known forms.

114. A corresponding transformation applies to the equation

$$\begin{vmatrix} 1, & \alpha, & \alpha', & \alpha\alpha' \\ 1, & \beta, & \beta', & \beta\beta' \\ 1, & \gamma, & \gamma', & \gamma\gamma' \\ 1, & \delta, & \delta', & \delta\delta' \end{vmatrix}=0,$$

which expresses the homography of two pairs. In fact, calling the determinant Ψ and representing by V the similar determinant

$$\begin{vmatrix} ss', & -s', & -s, & 1 \\ tt', & -t', & -t, & 1 \\ uu', & -u', & -u, & 1 \\ vv', & -v', & -v, & 1 \end{vmatrix},$$

which, equated to zero, would express the homography of the sets (s, t, u, v) and (s', t', u', v'), we have

$$V\Psi=\begin{vmatrix} (s-\alpha)(s'-\alpha'), & (s-\beta)(s'-\beta'), & (s-\gamma)(s'-\gamma'), & (s-\delta)(s'-\delta') \\ (t-\alpha)(t'-\alpha'), & (t-\beta)(t'-\beta'), & (t-\gamma)(t'-\gamma'), & (t-\delta)(t'-\delta') \\ (u-\alpha)(u'-\alpha'), & (u-\beta)(u'-\beta'), & (u-\gamma)(u'-\gamma'), & (u-\delta)(u'-\delta') \\ (v-\alpha)(v'-\alpha'), & (v-\beta)(v'-\beta'), & (v-\gamma)(v'-\gamma'), & (v-\delta)(v'-\delta') \end{vmatrix},$$

which gives various forms of the equation of homography. In particular, if $s=\alpha$, $s'=\beta'$, $t=\beta$, $t'=\alpha'$, $u=\gamma$, $u'=\delta'$, $v=\delta$, $v'=\gamma$, then

$$V\Psi=\begin{vmatrix} . & . & (\alpha-\gamma)(\beta'-\gamma'), & (\alpha-\delta)(\beta'-\delta') \\ . & . & (\beta-\gamma)(\alpha'-\gamma'), & (\beta-\delta)(\alpha'-\delta') \\ (\gamma-\alpha)(\delta'-\alpha'), & (\gamma-\beta)(\delta'-\beta') & . & . \\ (\delta-\alpha)(\gamma'-\alpha'), & (\delta-\beta)(\gamma'-\beta') & . & . \end{vmatrix},$$

and the right-hand side breaks up into factors, which are equal to each other (whence also $V = \Psi$), and the equation $\Psi = 0$ takes the form

$$(\alpha-\gamma)(\beta-\delta)(\alpha'-\delta')(\beta'-\gamma')-(\alpha-\delta)(\beta-\gamma)(\alpha'-\gamma')(\beta'-\delta')=0,$$

which is, in fact, one of the equations which express the equality of the anharmonic ratios of $(\alpha, \beta, \gamma, \delta)$ and $(\alpha', \beta', \gamma', \delta')$.

115. In the case of a cubic, the expressions considered are

$$(a, b, c, d \between x, y)^3, \tag{1}$$

$$(ac-b^2,\ ad-bc,\ bd-c^2 \between x, y)^2, \tag{2}$$

$$\left(\begin{array}{l} -a^2d+3abc-2b^3 \\ -abd+2ac^2-b^2c \\ +acd-2b^2d+bc^2 \\ +ad^2-3bcd+2c^3 \end{array}\between x, y\right)^3, \tag{3}$$

$$a^2d^2-6abcd+4ac^3+4b^3d-3b^2c^2, \tag{4}$$

where (1) is the cubic, (2) is the quadricovariant or Hessian, (3) is the cubicovariant, and (4) is the quartinvariant or discriminant.

And where it is convenient to do so, I write

$$(1) = U,$$
$$(2) = H,$$
$$(3) = \Phi,$$
$$(4) = \square,$$

so that we have

$$\Phi^2 - \square U^2 + 4H^3 = 0.$$

116. The Hessian may be written under the form

$$(ax+by)(cx+dy)-(bx+cy)^2,$$

(which, indeed, is the form under which *quà* Hessian it is originally given), and under the form

$$\begin{vmatrix} y^2, & -yx, & x^2 \\ a\,, & b\,, & c \\ b\,, & c\,, & d \end{vmatrix}.$$

The cubicovariant may be written under the form

$$\{2(ac-b^2)x+(ad-bc)y\}(bx^2+2cxy+dy^2)$$
$$-\{(ad-bc)x+2(bd-c^2)y\}(ax^2+2bxy+cy^2),$$

that is, as the Jacobian of the cubic and Hessian; and under the form

$$\tfrac{1}{2}(\partial_a, \partial_b, \partial_c, \partial_d \between y, -x)^3 \square,$$

that is, as the evectant of the discriminant.

The discriminant, taken negatively, may be written under the form

$$+4(ac-b^2)(bd-c^2)-(ad-bc)^2,$$

that is, as the discriminant of the Hessian.

117. We have

$$(a, b, c, d \between bx^2+2cxy+dy^2, -ax^2-2bxy-cy^2)^3 = U\Phi,$$

which expresses that a transmutant of the cubic is the product of the cubic and the cubicovariant. The equation

$$\{(\partial_a, \partial_b, \partial_c, \partial_d \between y, -x)^3\}^2 \square = 2U^2$$

expresses that the second evectant of the discriminant is the square of the cubic.

The equation

$$\begin{vmatrix} d^2, & -3cd, & -3bd+6c^2, & -3bc+2ad \\ -3cd, & -3c^3+12bd, & -3ad-6bc, & -3ac+6b^2 \\ -3bd+6c^2, & -3ad-6bc, & -3b^2+12ac, & 3ab \\ -3bc-12ad, & -3ac+6b^2, & 3ab, & a^2 \end{vmatrix} = 27\square^2$$

expresses that the determinant formed with the second differential coefficients of the discriminant gives the square of the discriminant.

The covariants of the intermediate $\alpha U+\beta\Phi$ are as follows, viz.

118. For the Hessian, we have

$$\begin{aligned} \tilde{H}(\alpha U+\beta\Phi) &= (1, 0, -\square \between \alpha, \beta)^2 H \\ &= (\alpha^2-\beta^2\square) H; \end{aligned}$$

for the cubicovariant,

$$\begin{aligned} \tilde{\Phi}(\alpha U+\beta\Phi) &= (0, \square, 0, -\square^2 \between \alpha, \beta)^3 U \\ &\quad + (1, 0, -\square, 0 \between \alpha, \beta)^3 \Phi \\ &= (\alpha^2-\beta^2\square)(\alpha\Phi+\beta\square U); \end{aligned}$$

and for the discriminant,

$$\begin{aligned} \tilde{\square}(\alpha U+\beta\Phi) &= (1, 0, -2\square, 0, \square^2 \between \alpha, \beta)^4 \Phi \\ &= (\alpha^2-\beta^2\square)^2 \square, \end{aligned}$$

where on the left-hand sides I have, for greater distinctness, written $\tilde{H}$, &c. to denote the functional operation of taking the Hessian, &c. of the operand $\alpha U+\beta\Phi$.

In particular, if $\alpha=0$, $\beta=1$,

$$\tilde{H}\Phi = -\square\,.\,H,$$

$$\tilde{\Phi}\Phi = -\square^2\,.\,U,$$

$$\tilde{\square}\Phi = \quad\square^3\,.$$

119. Solution of a cubic equation.

The question is to find a linear factor of the cubic

$$(a,\ b,\ c,\ d\between x,\ y)^3,$$

and this can be at once effected by means of the relation

$$\Phi^2 - \square U^2 = -4H^3$$

between the covariants. The equation in fact shows that each of the expressions

$$\tfrac{1}{2}(\Phi + U\sqrt{\square}),\quad \tfrac{1}{2}(\Phi - U\sqrt{\square})$$

is a perfect cube, and consequently that the cube root of each of these expressions is a linear function of (x, y). The expression

$$\sqrt[3]{\tfrac{1}{2}(\Phi + U\sqrt{\square})} + \sqrt[3]{\tfrac{1}{2}(\Phi - U\sqrt{\square})}$$

is consequently a linear function of x, y, and it vanishes when $U=0$, that is, the expression is a linear factor of the cubic.

It may be noticed here that the cubic being $a(x-\alpha y)(x-\beta y)(x-\gamma y)$, then we may write

$$\sqrt[3]{\tfrac{1}{2}(\Phi + U\sqrt{\square})} - \sqrt[3]{\tfrac{1}{2}(\Phi - U\sqrt{\square})} = \tfrac{1}{3}a(\omega-\omega^2)(\beta-\gamma)(x-\alpha y),$$

where ω is an imaginary cube root of unity: this will appear from the expressions which will be presently given for the covariants in terms of the roots.

120. Canonical form of the cubic.

The expressions $\tfrac{1}{2}(\Phi + U\sqrt{\square})$, $\tfrac{1}{2}(\Phi - U\sqrt{\square})$ are perfect cubes; and if we write

$$\tfrac{1}{2}(\Phi + U\sqrt{\square}) = \quad\sqrt{\square}\,\mathrm{x}^3,$$

$$\tfrac{1}{2}(\Phi - U\sqrt{\square}) = -\sqrt{\square}\,\mathrm{y}^3,$$

then we have

$$U = \quad \mathrm{x}^3 + \mathrm{y}^3,$$

$$\Phi = \sqrt{\square}\,(\mathrm{x}^3 - \mathrm{y}^3),$$

and thence also

$$H = -\sqrt[3]{\square}\,\mathrm{xy}.$$

121. When the cubic is expressed in terms of the roots, we have

$$a^{-1}U = (x-\alpha y)(x-\beta y)(x-\gamma y);$$

and then putting for shortness

$$A = (\beta-\gamma)(x-\alpha y),\quad B = (\gamma-\alpha)(x-\beta y),\quad C = (\alpha-\beta)(x-\gamma y),$$

so that

$$A+B+C=0,$$

we have

$$\begin{aligned} a^{-2}H &= -\tfrac{1}{18}(A^2+B^2+C^2) = \tfrac{1}{9}(BC+CA+AB),\\ a^{-3}\Phi &= -\tfrac{1}{27}(B-C)(C-A)(A-B),\\ a^{-4}\square &= -\tfrac{1}{27}(\beta-\gamma)^2(\gamma-\alpha)^2(\alpha-\beta)^2. \end{aligned}$$

122. The covariants H, Φ are most simply expressed as above, but it may be proper to add the equations

$$\begin{aligned} a^{-2}H &= -\tfrac{1}{18}\Sigma(\beta-\gamma)^2(x-\alpha y)^2\\ &= -\tfrac{1}{9}\left\{\begin{array}{l} \alpha^2+\beta^2+\gamma^2-\beta\gamma-\gamma\alpha-\alpha\beta,\\ 6\alpha\beta\gamma-\beta\gamma^2-\gamma\alpha^2-\alpha\beta^2-\beta^2\gamma-\gamma^2\alpha-\alpha^2\beta,\\ \beta^2\gamma^2+\gamma^2\alpha^2+\alpha^2\beta^2-\alpha^2\beta\gamma-\beta^2\gamma\alpha-\gamma^2\alpha\beta \end{array}\right\}(x,\ y)^2\\ &= -\tfrac{1}{9}\{(\alpha+\omega\beta+\omega^2\gamma)x+(\beta\gamma+\omega\gamma\alpha+\omega^2\alpha\beta)y\}\{(\alpha+\omega^2\beta+\omega\gamma)x+(\beta\gamma+\omega^2\gamma\alpha+\omega\alpha\beta)y\} \end{aligned}$$

(where ω is an imaginary cube root of unity),

$$\begin{aligned} a^{-3}\Phi &= \tfrac{1}{27}\Sigma(\alpha-\beta)(\alpha-\gamma)^2(x-\beta y)^2(x-\gamma y)\\ &= \left\{\begin{array}{l} 2(\alpha^3+\beta^3+\gamma^3)-3(\beta\gamma^2+\gamma\alpha^2+\alpha\beta^2+\beta^2\gamma+\gamma^2\alpha+\alpha^2\beta)+12\alpha\beta\gamma,\\ -2(\alpha^2\beta\gamma+\beta^2\gamma\alpha+\gamma^2\alpha\beta)+4(\beta^2\gamma^2+\gamma^2\alpha^2+\alpha^2\beta^2)-(\beta\gamma^3+\gamma\alpha^3+\alpha\beta^3+\beta^3\gamma+\gamma^3\alpha+\alpha^3\beta),\\ -2(\alpha\beta^2\gamma^2+\beta\gamma^2\alpha^2+\gamma\alpha^2\beta^2)+4(\alpha^3\beta\gamma+\beta^3\gamma\alpha+\gamma^3\alpha\beta)-(\beta^2\gamma^3+\gamma^2\alpha^3+\alpha^2\beta^3+\beta^3\gamma^2+\gamma^3\alpha^2+\alpha^3\beta^2),\\ +2(\beta^3\gamma^3+\gamma^3\alpha^3+\alpha^3\beta^3)-3(\alpha\beta^2\gamma^3+\beta\gamma^2\alpha^3+\gamma\alpha^2\beta^3+\alpha\beta^3\gamma^2+\beta\gamma^3\alpha^2+\gamma\alpha^2\beta^3)+12\alpha^2\beta^2\gamma^2 \end{array}\right\}(x,\ y)^3\\ &= \{(2\alpha-\beta-\gamma)x+(2\beta\gamma-\gamma\alpha-\alpha\beta)y\}\{(2\beta-\gamma-\alpha)x+(2\gamma\alpha-\alpha\beta-\beta\gamma)y\}\{(2\gamma-\alpha-\beta)x+(2\alpha\beta-\beta\gamma-\gamma\alpha)y \end{aligned}$$

123. It may be observed that we have $a^{-6}\square U^2 = -\frac{1}{27}A^2B^2C^2$, which, with the above values of H, Φ in terms of A, B, C and the equation $A+B+C=0$, verifies the equation $\Phi^2-\square U^2+4H^3=0$, which connects the covariants. In fact, we have identically,

$$\begin{aligned} &(B-C)^2(C-A)^2(A-B)^2 =\\ &-4(A+B+C)^3ABC+(A+B+C)^2(BC+CA+AB)^2+18(A+B+C)(BC+CA+AB)ABC\\ &\qquad -4(BC+CA+AB)^3-27A^2B^2C^2, \end{aligned}$$

by means of which the verification can be at once effected.

124. If, as before, ω is an imaginary cube root of unity, then we may write

$$27a^{-3}\Phi \quad = -(B-C)(C-A)(A-B),$$

$$27a^{-3}U\sqrt{\square} = \quad 3(\omega-\omega^2)ABC,$$

and these values give

$$27a^{-3}\tfrac{1}{2}(\Phi+U\sqrt{\square} = \{(\alpha+\omega^2\beta+\omega\gamma)x+(\beta\gamma+\omega^2\gamma\alpha+\omega\alpha\beta)y\}^3,$$

$$27a^{-3}\tfrac{1}{2}(\Phi-U\sqrt{\square} = \{(\alpha+\omega\beta+\omega^2\gamma)x+(\beta\gamma+\omega\gamma\alpha+\omega^2\alpha\beta)y\}^3,$$

and we thence obtain

$$\sqrt[3]{\tfrac{1}{2}(\Phi+U\sqrt{\square})}-\sqrt[3]{\tfrac{1}{2}(\Phi-U\sqrt{\square})} = -\tfrac{1}{3}a(\omega-\omega^2)(\beta-\gamma)(x-\alpha y),$$

which agrees with a former result.

125. The preceding formulæ show without difficulty, that each factor of the cubicovariant is the harmonic of a factor of the cubic with respect to the other two factors of the cubic; and moreover, that the factors of the cubic and the cubicovariant form together an involution having for sibiconjugates the factors of the Hessian. In fact, the harmonic of $x-\alpha y$ with respect to $(x-\beta y)(x-\gamma y)$ is $(2\alpha-\beta-\gamma)x+(2\beta\gamma-\gamma\alpha-\alpha\beta)y$, which is a factor of the cubicovariant; the product of the pair of harmonic factors is

$$(2\alpha-\beta-\gamma)x^2+2(\beta\gamma-\alpha^2)xy+(-2\alpha\beta\gamma+\alpha^2\beta+\alpha^2\gamma)y^2;$$

and multiplying this by $\beta-\gamma$, and taking the sum of the analogous expressions, this sum vanishes, or the three pairs form an involution. That the Hessian gives the sibiconjugates of the involution is most readily shown as follows:—the last-mentioned quadric may be written

$$\big(-(\alpha+\beta+\gamma)+3\alpha\big)x^2+2\big(\alpha\beta+\alpha\gamma+\beta\gamma-\alpha(\alpha+\beta+\gamma)\big)xy+\big(-3\alpha\beta\gamma+\alpha(\alpha\beta+\alpha\gamma+\beta\gamma)\big)y^2,$$

which is equal to

$$\left(3\frac{b}{a}+3\alpha\right)x^2+2\left(3\frac{c}{a}-3\frac{b}{a}\alpha\right)xy+\left(3\frac{d}{a}+3\frac{c}{a}\alpha\right)y^2,$$

or, throwing out the factor $3a^{-1}$, to

$$(b+a\alpha,\quad 2c-2b\alpha,\quad d+c\alpha\between x,\ y)^2,$$

which is harmonically related to the Hessian

$$(ac-b^2,\quad ad-bc,\quad bd-c^2\between x,\ y)^2;$$

and in like manner the other two pairs of factors will be also harmonically related to the Hessian.

126. In the case of a pair of equal roots, we have

$$a^{-1}U = (x-\alpha y)^2(x-\gamma y),$$
$$a^{-2}H = -\tfrac{1}{9}(\alpha-\gamma)^2(x-\alpha y)^2,$$
$$a^{-3}\Phi = -\tfrac{2}{27}(\alpha-\gamma)^3(x-\alpha y)^3,$$
$$\square = 0.$$

And in the case of all the roots equal, we have

$$a^{-1}U = (x-\alpha y)^3,$$
$$H=0,\quad \Phi=0,\quad \square=0.$$

127. In the solution of a biquadratic equation we have to consider the cubic equation $\varpi^3 - M(\varpi - 1) = 0$. The cubic here is $(1, 0, -M, M \between \varpi, 1)^3$, or what is the same thing,

$$(1, 0, -\tfrac{1}{3}M, M \between \varpi, 1)^3;$$

the Hessian is

$$M(-\tfrac{1}{3}, 1, -\tfrac{1}{9}M \between \varpi, 1)^2;$$

the cubicovariant is

$$M(-1, \tfrac{2}{3}M, -\tfrac{1}{3}M, M+\tfrac{2}{27}M^2 \between \varpi, 1)^3;$$

and the discriminant is

$$M^2(1-\tfrac{4}{27}M).$$

128. In the case of a quartic, the expressions considered are

$$(a, b, c, d, e \between x, y)^4, \tag{1}$$

$$ae - 4bd + 3c^2, \tag{2}$$

$$(ac-b^2,\ 2(ad-bc),\ ae+2bd-3c^2,\ 2(be-cd),\ ce-d^2 \between x,\ y)^4, \tag{3}$$

$$ace + 2bcd - ad^2 - b^2e - c^3, \tag{4}$$

$$\left\{\begin{array}{l} -\ a^2d + 3\,abc - 2\,b^3, \\ -\ a^2e - 2\,abd + 9\,ac^2 - 6\,b^2c, \\ -\ 5\,abe + 15\,acd - 10\,b^2d, \\ +10\,ad^2 - 10\,b^2e, \\ +\ 5\,ade + 10\,bd^2 - 15\,bce, \\ +\ ae^2 + 2\,bde - 9\,c^2e + 6\,cd^2, \\ +\ be^2 - 3\,cde + 2\,d^3 \end{array}\right\} \between x,\ y)^6, \tag{5}$$

where (1) is the quartic, (2) is the quadrinvariant, (3) is the quadricovariant or Hessian, (4) is the cubinvariant, and (5) is the cubicovariant.

And where it is convenient to do so, I write

$$(1) = U,$$
$$(2) = I,$$
$$(3) = H,$$
$$(4) = J,$$
$$(5) = \Phi.$$

The preceding covariants are connected by the equation

$$JU^3 - IU^2H + 4H^3 = -\Phi^2.$$

The discriminant is not an irreducible invariant, its value is

$$\square = I^3 - 27J^2 = a^3e^3 + \&c.,$$

for which see Table No. 12, [p. 272].

129. It is for some purposes convenient to arrange the expanded expression of the discriminant in powers of the middle coefficient c. We thus have

$$\begin{aligned}\square = \; & a^3e^3 - 12\,a^2bde^2 - 27\,a^2d^4 - 6\,ab^2d^2e - 27\,b^4e^2 - 64\,b^3d^3 \\ & + c\,(\,54\,a^2d^2e + 54\,ab^2e^2 + 108\,abd^3 + 108\,b^3de) \\ & + c^2\,(-18\,a^2e^2 - 180\,abde + 36\,b^2d^2) \\ & + c^3\,(-54\,ad^2 - 54\,b^2e) \\ & + c^4\,(81\,ae).\end{aligned}$$

130. Solution of a biquadratic equation.

We have to find a linear factor of the quartic

$$(a,\, b,\, c,\, d,\, e \between x,\, y)^4.$$

The equation $JU^3 - IU^2H + 4H^3 = -\Phi^2$, putting for shortness

$$M = \frac{I^3}{4J^2},$$

may be written

$$(1,\ 0,\ -M,\ M \between IH,\ JU)^3 = -\tfrac{1}{4}I^3\Phi^2.$$

Hence, if ϖ_1, ϖ_2, ϖ_3 are the roots of

$$(1,\ 0,\ -M,\ M \between \varpi,\ 1)^3 = 0,$$

the expressions $IH - \varpi_1 JU$, $IH - \varpi_2 JU$, $IH - \varpi_3 JU$ are each of them squares; write

$$(\varpi_2 - \varpi_3)(IH - \varpi_1 JU) = X^2,$$
$$(\varpi_3 - \varpi_1)(IH - \varpi_2 JU) = Y^2,$$
$$(\varpi_1 - \varpi_2)(IH - \varpi_3 JU) = Z^2,$$

so that, identically,

$$X^2 + Y^2 + Z^2 = 0\,;$$

and consequently $X + \iota Y$, $X - \iota Y$ are each of them squares. The expression

$$\alpha X + \beta Y + \gamma Z$$

will be a square if only

$$\alpha^2 + \beta^2 + \gamma^2 = 0,$$

as may be seen by writing it under the form

$$\tfrac{1}{2}(\alpha + \iota\beta)(X - \iota Y) + \tfrac{1}{2}(\alpha - \iota\beta)(X + \iota Y) - \gamma i \sqrt{X^2 + Y^2}\,;$$

and in particular, writing $\sqrt{\varpi_2 - \varpi_3}$, $\sqrt{\varpi_3 - \varpi_1}$, $\sqrt{\varpi_1 - \varpi_2}$ for α, β, γ, the expression

$$(\varpi_2 - \varpi_3)\sqrt{IH - \varpi_1 JU} + (\varpi_3 - \varpi_1)\sqrt{IH - \varpi_2 JU} + (\varpi_1 - \varpi_2)\sqrt{IH - \varpi_3 JU}$$

is a square; and since the product of the different values is a multiple of U^2 (this is most readily perceived by observing that the expression vanishes for $U = 0$), the expression is the square of a linear factor of the quartic.

131. To complete the solution: ϖ_1, ϖ_2, ϖ_3 are the roots of the cubic equation

$$(1,\ 0,\ -\tfrac{1}{3}M,\ M \between \varpi,\ 1)^3 = 0\,;$$

and hence, putting for shortness,

$$P^3 = \tfrac{1}{2}M\,\{(-1,\ \tfrac{2}{9}M,\ -\tfrac{1}{3}M,\ M + \tfrac{2}{27}M^2 \between IH,\ JU)^3 + \sqrt{1 - \tfrac{4}{27}M}\,(1,\ 0,\ -\tfrac{1}{3}M,\ M \between IH,\ JU)^3,$$

$$Q^3 = \tfrac{1}{2}M\,\{(-1,\ \tfrac{2}{9}M,\ -\tfrac{1}{3}M,\ M + \tfrac{2}{27}M^2 \between IH,\ JU)^3 - \sqrt{1 - \tfrac{4}{27}M}\,(1,\ 0,\ -\tfrac{1}{3}M,\ M \between IH,\ JU)^3,$$

we have (ω being an imaginary cube root of unity)

$$\tfrac{1}{3}(\omega - \omega^2)(\varpi_2 - \varpi_3)(IH - \varpi_1 JU) = P - Q\,;$$

and if

$$P_0{}^3 = \tfrac{1}{2}M\,\{-1 + \sqrt{1 - \tfrac{4}{27}M}\},$$

$$Q_0{}^3 = \tfrac{1}{2}M\,\{-1 - \sqrt{1 - \tfrac{4}{27}M}\},$$

then

$$\tfrac{1}{3}(\omega - \omega^2)(\varpi_2 - \varpi_3) = P_0 - Q_0.$$

Hence, multiplying and observing that $(\omega - \omega^2)^2 = -3$, we find

$$-\frac{1}{(\omega - \omega^2)^2}(\varpi_2 - \varpi_3)^2(IH - \varpi_1 JU) = (P - Q)(P_0 - Q_0),$$

and consequently

$$(\varpi_2 - \varpi_3)\sqrt{IH - \varpi_1 JU} = (\omega - \omega^2)\sqrt{-(P - Q)(P_0 - Q_0)}.$$

We have, in like manner,

$$\tfrac{1}{3}(\omega - \omega^2)(\varpi_2 - \varpi_3)(IH - \varpi_1 JU) = P - Q,$$

$$\tfrac{1}{3}(\omega - \omega^2)(\varpi_3 - \varpi_1)(IH - \varpi_2 JU) = \omega P - \omega^2 Q,$$

$$\tfrac{1}{3}(\omega - \omega^2)(\varpi_1 - \varpi_2)(IH - \varpi_3 JU) = \omega^2 P - \omega Q,$$

and
$$\tfrac{1}{3}(\omega-\omega^2)(\varpi_2-\varpi_3)=P_0-\ \ Q_0,$$
$$\tfrac{1}{3}(\omega-\omega^2)(\varpi_3-\varpi_1)=\omega\,P_0-\omega^2Q_0,$$
$$\tfrac{1}{3}(\omega-\omega^2)(\varpi_1-\varpi_2)=\omega^2P_0-\omega\,Q_0,$$

and therefore
$$(\varpi_2-\varpi_3)\sqrt{IH-\varpi_1JU}=(\omega-\omega^2)\sqrt{-\ \ (P-Q)\ \ \ \ (P_0-Q_0)},$$
$$(\varpi_3-\varpi_1)\sqrt{IH-\varpi_2JU}=(\omega-\omega^2)\sqrt{-(\omega P-\omega^2Q)(\omega P_0-\omega^2Q)},$$
$$(\varpi_3-\varpi_1)\sqrt{IH-\varpi_3JU}=(\omega-\omega^2)\sqrt{-(\omega^2P-\omega Q)(\omega^2P_0-\omega Q_0)};$$

and hence disregarding the common factor $\omega-\omega^2$, the square of the linear factor of the quartic is
$$\sqrt{-(P-Q)(P_0-Q_0)}+\sqrt{-(\omega P-\omega^2Q)(\omega P_0-\omega^2Q_0)}+\sqrt{-(\omega^2P-\omega Q)(\omega^2P_0-\omega Q_0)},$$

which is the required solution.

It may be proper to add that
$$-\varpi_1=\ \ P_0+\ \ Q_0,$$
$$-\varpi_2=\omega\,P_0+\omega^2Q_0,$$
$$-\varpi_3=\omega^2P_0+\omega\,Q_0.$$

132. The solution gives at once the canonical form of the quartic; in fact, writing
$$X+\iota Y=2\sqrt{(\varpi_2-\varpi_3)(\varpi_3-\varpi_1)}\sqrt{J}\,\mathrm{x}^2,$$
$$X-\iota Y=2\sqrt{(\varpi_2-\varpi_3)(\varpi_3-\varpi_1)}\sqrt{J}\,\mathrm{y}^2,$$

where X, Y have their former significations, we find, by a simple reduction,
$$IH-\varpi_1JU=\ \ (\varpi_3-\varpi_1)\,J\,(\mathrm{x}^2+\mathrm{y}^2)^2,$$
$$IH-\varpi_2JU=-(\varpi_2-\varpi_3)\,J\,(\mathrm{x}^2-\mathrm{y}^2)^2,$$
$$IH-\varpi_3JU=-\frac{(\varpi_2-\varpi_3)(\varpi_3-\varpi_1)}{\varpi_1-\varpi_2}\,J\,.\,4\mathrm{x}^2\mathrm{y}^2,$$

and thence putting
$$\theta=-\frac{\varpi_2}{\varpi_1-\varpi_2}=\frac{\tfrac{1}{3}(\omega-\omega^2)(\omega^2P_0+\omega Q_0)}{(\omega^2P_0-\omega Q_0)},$$

we have
$$U=\mathrm{x}^4+\mathrm{y}^4+6\theta\mathrm{x}^2\mathrm{y}^2,$$

which is the form required.

133. The Hessian may be written under the form
$$(\partial_e,\ -\partial_d,\ \ \partial_c,\ -\partial_b,\ \ \partial_a\between x,y)^4J,$$

that is, as the evectant of the cubinvariant.

The cubicovariant may be obtained by writing the quartic under the form

$$(ax+by,\ bx+cy,\ cx+dy,\ dx+ey \between x,\ y)^3,$$

and then, treating the linear functions as coefficients, or considering this as a cubic, the cubicovariant of the cubic gives the cubicovariant of the quartic.

If we represent the cubicovariant by

$$\Phi = (\mathrm{a},\ \mathrm{b},\ \mathrm{c},\ \mathrm{d},\ \mathrm{e},\ \mathrm{f},\ \mathrm{g} \between x,\ y)^6,$$

then we have identically,

$$\mathrm{ag} - 9\mathrm{ce} + 8\mathrm{d}^2 = 0\,;$$

and moreover forming the quadrinavariant of the sextic, we find

$$\mathrm{ag} - 6\mathrm{bf} + 15\mathrm{ce} - 10\mathrm{d}^2 = \tfrac{1}{6}\square,$$

where $\square$ is the discriminant of the quartic. From these two equations we find

$$\mathrm{bf} - 4\mathrm{ce} + 3\mathrm{d}^2 = -\tfrac{1}{36}\square,$$

which is an expression given by Mr Salmon: it is the more remarkable as the left-hand side is the quadrinvariant of $(\mathrm{b},\ \mathrm{c},\ \mathrm{d},\ \mathrm{e},\ \mathrm{f} \between x,\ y)^4$, which is *not* a covariant of the quartic. It may be noticed also that we have

$$\mathrm{af} - 3\mathrm{be} + 2\mathrm{cd} = 0,$$
$$\mathrm{bg} - 3\mathrm{cf} + 2\mathrm{de} = 0.$$

134. The covariants of the intermediate

$$\alpha U + 6\beta H$$

of the quartic and Hessian are as follows, viz.

The quadrinvariant is

$$\tilde{I}\,(\alpha U + 6\beta H) = \ (I,\ 18J,\ 3I^2 \between \alpha,\ \beta)^2\,;$$

the cubinvariant is

$$\tilde{J}\,(\alpha U + 6\beta H) = \ (J,\ I^2,\ 9IJ,\ -I^3 + 54J^2 \between \alpha,\ \beta)^3\,;$$

the Hessian is

$$\tilde{H}\,(\alpha U + 6\beta H) = \ (1,\ 0,\ -3I \between \alpha,\ \beta)^2\, H$$
$$+ (0,\ I,\ \ 9J \between \alpha,\ \beta)^2\, U\,;$$

and the cubicovariant is

$$\tilde{\Phi}\,(\alpha U + 6\beta H) = \ (1,\ 0,\ -9I,\ -54J \between \alpha,\ \beta)^3\, \Phi\,;$$

to which may be added the discriminant, which is

$$\tilde{\square}\,(\alpha U + 6\beta H) = (1,\ 0,\ -18I,\ 108J,\ 81I^2,\ 972IJ,\ -2916J^2 \between \alpha,\ \beta)^6\, \square.$$

135. The expression for the lambdaic is

$$\begin{vmatrix} a \;, & b \;, & c-2\lambda \\ b \;, & c+\lambda, & d \\ c-2\lambda, & d \;, & e \end{vmatrix} = J + \lambda I - 4\lambda^3.$$

If the determinant is represented by Λ, that is if

$$\Lambda = -4\lambda^3 + \lambda I + J,$$

then if λ_1, λ_2, λ_3 are the roots of the equation $\Lambda = 0$, and if the values of $\partial_a \Lambda$, &c. obtained by writing λ_1 in the place of λ are represented by $\partial_a \Lambda_1$, &c., then if x, y satisfy the equation

$$(a, b, c, d, e \between x, y)^4 = 0,$$

we have identically (X, Y being arbitrary),

$$\frac{(a, b, c, d, e \between X, Y \overset{3}{\between} x, y)}{Xy - Yx}$$

$$= \sqrt{-(\partial_e, -\partial_d, \partial_c, -\partial_b, \partial_a \between X, Y)^4 \Lambda_1}$$

$$+ \sqrt{-(\partial_e, -\partial_d, \partial_c, -\partial_b, \partial_a \between X, Y)^4 \Lambda_2}$$

$$+ \sqrt{-(\partial_e, -\partial_d, \partial_c, -\partial_b, \partial_a \between X, Y)^4 \Lambda_3}$$

a theorem due to Aronhold. I have quoted this theorem in its original form as an application of the lambdaic, but I remark that

$$-(\partial_e, -\partial_d, \partial_c, -\partial_b, \partial_a \between X, Y)^4 \Lambda = -\lambda (a, \ldots \between X, Y)^4 - (ac - b^2, \ldots \between X, Y)^4 = -\lambda U' - H'$$

if U', H' are what U, H become, substituting for (x, y) the new facients (X, Y). Moreover, we have

$$\lambda = -\frac{J\varpi}{I};$$

for substituting this value in the equation $\Lambda = 0$, we obtain the before-mentioned equation $\varpi^3 - M(\varpi - 1) = 0$. We have, therefore,

$$-(\partial_e, -\partial_d, \partial_c, -\partial_b, \partial_a \between X, Y)^4 \Lambda = \frac{J\varpi}{I} U' - H' = -\frac{1}{I}(IH' - J\varpi U'),$$

and the equation becomes

$$\frac{(a, b, c, d, e \between X, Y \overset{3}{\between} x, y)}{Xy - Yx} \sqrt{-I} = \sqrt{IH' - J\varpi_1 U'} + \sqrt{IH' - J\varpi_2 U'} + \sqrt{IH' - J\varpi_3 U'}.$$

Moreover, if $(x - \alpha y)$ be a factor of the quartic, then replacing in the formula y by the value αx, (x, y) will disappear altogether; and then changing (X, Y) into (x, y) where x, y are now arbitrary, we have

$$\frac{(a, b, c, d, e \between x, y)^3 (\alpha, 1)}{x - \alpha y} \sqrt{-I} = \sqrt{IH - \varpi_1 JU} + \sqrt{IH - \varpi_2 JU} + \sqrt{IH - \varpi_3 JU},$$

which is a form connected with the results in Nos. 130 and 131.

136. We have

$$\begin{vmatrix} y^4, & -4xy^3, & 6x^2y^2, & -4x^3y, & x^4 \\ & a\ , & 3b\ , & 3c\ , & d \\ a\ , & 3b\ , & 3c\ , & d\ , & \\ & b\ , & 3c\ , & 3d\ , & e \\ b\ , & 3c\ , & 3d\ , & e\ , & \end{vmatrix} = 6IH - 9JU\,;$$

it will appear from the formulæ relating to the roots of the quartic, that the expression $6IH - 9JU$ vanishes identically when there are two pairs of equal roots, or what is the same thing, when the quartic is a perfect square. The conditions in order that the expression may vanish are obviously

$$\begin{array}{ccccccccccc} 6(ac - b^2) & : & 3(ad - bc) & : & ae + 2bd - 3c^2 & : & 3(be - cd) & : & 6(ce - d^2) & : & 9J \\ = a & : & b & : & c & : & d & : & e & : & I, \end{array}$$

conditions which imply that the several determinants

$$\left\| \begin{matrix} 6(ac - b^2), & 3(ad - bc), & ae + 2bd - 3c^2, & 3(be - cd), & 6(ce - d^2) \\ a & b & c & d & e \end{matrix} \right\|,$$

all of them vanish. If for a moment we write $6H = (a', b', c', d', e' \between x, y)^4$, then the determinants are

$$\left\| \begin{matrix} a', & b', & c', & d', & e' \\ a, & b, & c, & d, & e \end{matrix} \right\|;$$

we have identically

$$\begin{aligned} ad' - a'd &= 3(bc' - b'c), \\ eb' - e'b &= 3(dc' - d'c), \\ ae' - a'e &= 3(bd' - b'd), \end{aligned}$$

and the ten determinants thus reduce themselves to seven determinants only, these in fact being, to mere numerical factors *près*, the coefficients of the cubicovariant; this perfectly agrees with a subsequent result, viz. that the cubicovariant vanishes identically when the quartic is a perfect square.

137. It may be remarked that the equation $6IH - 9JU = 0$ will be satisfied identically if

$$a = \frac{b^2}{c-\phi},\quad e = \frac{d^2}{c-\phi},\quad bd = (c-\phi)(c+2\phi),$$

where ϕ is arbitrary; the quartic is in this case the square of

$$\left(\frac{b}{\sqrt{c-\phi}},\ \sqrt{c-\phi},\ \frac{d}{\sqrt{c-\phi}}\right)(x,\ y)^2.$$

If with the conditions in question we combine the equation $I = 0$ (which in this case implies also $J = 0$), we obtain $\phi = 0$, and consequently

$$\frac{a}{b} = \frac{b}{c} = \frac{c}{d} = \frac{d}{e},$$

or the quartic will be a complete fourth power.

It is easy to express in terms of the coefficients a', b', c', d', e' of $6H$ the different determinants

$$\left\| \begin{matrix} a, & b, & c, & d \\ b, & c, & d, & e \end{matrix} \right\|,$$

we have in fact

$$\left\{ \begin{aligned} ae - bd &= \tfrac{1}{2}\left(c' + \frac{1}{\sqrt{3}}\sqrt{a'e' + 4b'd' - 3c'^2}\right), \\ 3(bd - c^2) &= \tfrac{1}{2}\left(c' - \frac{1}{\sqrt{3}}\sqrt{a'e' + 4b'd' - 3c'^2}\right), \\ ac - b^2 &= \tfrac{1}{6}a', \\ ad - bc &= \tfrac{1}{3}b', \\ be - cd &= \tfrac{1}{3}c', \\ ce - d^2 &= \tfrac{1}{6}e', \end{aligned} \right.$$

whence all the above-mentioned determinants will vanish, or the quartic will be a perfect fourth power if only the Hessian vanishes identically.

138. Considering the quartic as expressed in terms of the roots, we have

$$a^{-1}U = (x - \alpha y)(x - \beta y)(x - \gamma y)(x - \delta y);$$

and if we write for shortness

$$\begin{aligned} A &= (\beta - \gamma)(\alpha - \delta), \\ B &= (\gamma - \alpha)(\beta - \delta), \\ C &= (\alpha - \beta)(\gamma - \delta), \end{aligned}$$

which are connected by

$$A + B + C = 0,$$

then we have

$$a^{-2} I = \tfrac{1}{24} (A^2 + B^2 + C^2) = -\tfrac{1}{12} (BC + CA + AB),$$
$$a^{-3} J = \tfrac{1}{432} (B - C)(C - A)(A - B);$$

and for the discriminant we have

$$a^{-6} \square = \tfrac{1}{256} (\alpha - \beta)^2 (\alpha - \gamma)^2 (\alpha - \delta)^2 (\beta - \gamma)^2 (\beta - \delta)^2 (\gamma - \delta)^2,$$
$$= \tfrac{1}{256} A^2 B^2 C^2,$$

and it is easy by means of a preceding formula to verify the equation $\square = I^3 - 27J^2$.

139. The formulæ show a very remarkable analogy between the covariants of a cubic and the invariants of a quartic. In fact

For the cubic.	For the quartic.
$\begin{cases} A = (\beta - \gamma)(x - \alpha y), \\ B = (\gamma - \alpha)(x - \beta y), \\ C = (\alpha - \beta)(x - \gamma y), \end{cases}$	$\begin{cases} A = (\beta - \gamma)(\alpha - \delta), \\ B = (\gamma - \alpha)(\beta - \delta), \\ C = (\alpha - \beta)(\gamma - \delta); \end{cases}$

and then we have corresponding to each other:

For the cubic.	For the quartic.
The Hessian,	The quadrinvariant,
The cubicovariant,	The cubinvariant,
The cubic into the square root of the discriminant.	The discriminant.

140. For the two covariants, we have

$$a^{-2} H = -\tfrac{1}{48} \Sigma (\alpha - \beta)^2 (x - \gamma y)^2 (x - \delta y)^2,$$

and

$$a^{-3} \Phi = -\tfrac{1}{32} \mathfrak{A}\mathfrak{B}\mathfrak{C},$$

if for shortness,

$$\mathfrak{A} = (\delta + \alpha - \beta - \gamma,\ \ -\delta\alpha + \beta\gamma,\ \ \delta\alpha\,(\beta + \gamma) - \beta\gamma\,(\delta + \alpha) \between x, y)^2,$$
$$\mathfrak{B} = (\delta + \beta - \gamma - \alpha,\ \ -\delta\beta + \gamma\alpha,\ \ \delta\beta\,(\gamma + \alpha) - \gamma\alpha\,(\delta + \beta) \between x, y)^2,$$
$$\mathfrak{C} = (\delta + \gamma - \alpha - \beta,\ \ -\delta\gamma + \alpha\beta,\ \ \delta\gamma\,(\alpha + \beta) - \alpha\beta\,(\delta + \gamma) \between x, y)^2.$$

141. We have

$$M = \tfrac{27}{8} \frac{(A^2 + B^2 + C^2)^3}{(B - C)^2 (C - A)^2 (A - B)^2};$$

or putting for shortness

$$\Lambda = \tfrac{3}{2} \frac{A^2 + B^2 + C^2}{(B - C)(C - A)(A - B)},$$

we have

$$M = \tfrac{3}{2} (A^2 + B^2 + C^2)\, \Lambda^2;$$

and it is then easy to deduce

$$\varpi_1 = \Lambda(B - C),$$
$$\varpi_2 = \Lambda(C - A),$$
$$\varpi_3 = \Lambda(A - B);$$

in fact, these values give

$$\begin{aligned}\varpi_1 + \varpi_2 + \varpi_3 &= 0,\\ \varpi_1\varpi_2 + \varpi_1\varpi_3 + \varpi_2\varpi_3 &= -M,\\ \varpi_1\varpi_2\varpi_3 &= M,\end{aligned}$$

and they are consequently the roots of the equation $\varpi^3 - M(\varpi - 1) = 0$.

142. The leading coefficient of $IH - \varpi_1 JU$ is then equal to a^4 into the following expression, viz.

$$\tfrac{1}{24}(A^2 + B^2 + C^2)\, a^{-2}(ac - b^2) - \tfrac{1}{288}(A^2 + B^2 + C^2)(B - C),$$

which is equal to

$$\tfrac{1}{1152}(A^2 + B^2 + C^2)\{48a^{-2}(ac - b^2) - 4(B - C)\},$$

and the term in { } is

$$8(\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta) - 3(\alpha + \beta + \gamma + \delta)^2 - 4(\gamma - \alpha)(\beta - \delta) + 4(\alpha - \beta)(\gamma - \delta),$$

which is equal to

$$-3(\delta + \alpha - \beta - \gamma)^2.$$

But $IH - \varpi_1 JU$ is a square, and it is easy to complete the expression, and we have

$$a^{-4}(IH - \varpi_1 JU) = -\tfrac{1}{384}(A^2 + B^2 + C^2)\{(\delta + \alpha - \beta - \gamma,\ -\delta\alpha + \beta\gamma,\ \delta\alpha(\beta + \gamma) - \beta\gamma(\delta + \alpha)\between x, y)^2\}^2$$
$$a^{-4}(IH - \varpi_2 JU) = -\tfrac{1}{384}(A^2 + B^2 + C^2)\{(\delta + \beta - \gamma - \alpha,\ -\delta\beta + \gamma\alpha,\ \delta\beta(\gamma + \alpha) - \gamma\alpha(\delta + \beta)\between x, y)^2\}^2,$$
$$a^{-4}(IH - \varpi_3 JU) = -\tfrac{1}{384}(A^2 + B^2 + C^2)\{(\delta + \gamma - \alpha - \beta,\ -\delta\gamma + \alpha\beta,\ \delta\gamma(\alpha + \beta) - \alpha\beta(\gamma + \delta)\between x, y)^2\}^2.$$

We have, moveover,

$$\varpi_2 - \varpi_3 = -3\Lambda A,$$
$$\varpi_3 - \varpi_1 = -3\Lambda B,$$
$$\varpi_1 - \varpi_2 = -3\Lambda C,$$

and thence

$$a^{-2}(\varpi_2 - \varpi_3)\sqrt{IH - \varpi_1 JU} = \tfrac{1}{8}(\omega - \omega^2)\frac{A^2 + B^2 + C^2}{(B - C)(C - A)(A - B)}(\beta - \gamma)(\alpha - \delta)$$
$$\times(\delta + \alpha - \beta - \gamma,\quad -\delta\alpha + \beta\gamma,\quad \delta\alpha(\beta + \gamma) - \beta\gamma(\delta + \alpha)\between x, y)^2;$$

and taking the sum of the analogous expressions, we find

$$a^{-2}\{(\varpi_2 - \varpi_3)\sqrt{IH - \varpi_1 JU} + (\varpi_3 - \varpi_1)\sqrt{IH - \varpi_2 JU} + (\varpi_1 - \varpi_2)\sqrt{IH - \varpi_3 JU}\}$$
$$= -\tfrac{1}{4}(\omega - \omega^2)\frac{A^2 + B^2 + C^2}{(B - C)(C - A)(A - B)}(\alpha - \beta)(\beta - \gamma)(\gamma - \alpha)(x - \delta y)^2,$$

which agrees with a former result.

143. The equation $I = 0$ gives

$$A : B : C = 1 : \omega : \omega^2,$$

where ω is an imaginary cube root of unity; the factors of the quartic may be said in this case to be *Symmetric Harmonics*.

The equation $J = 0$ gives one of the three equations,

$$A = B, \quad B = C, \quad C = A;$$

in this case a pair of factors of the quartic are harmonics with respect to the other pair of factors. If we have simultaneously $I = 0$, $J = 0$, then

$$A = B = C = 0,$$

and in this case three of the factors of the quartic are equal.

144. If any two of the linear factors of the quartic are considered as forming, with the other two linear factors, an involution, the sibiconjugates of the involution make up a quadratic factor of the cubicovariant; and considering the three pairs of sibiconjugates, or what is the same thing, the six linear factors of the cubicovariant, the factors of a pair are the sibiconjugates of the involution formed by the other two pairs of factors.

In fact, the sibiconjugates of the involution formed by the equations

$$(x - \alpha y)(x - \delta y) = 0, \quad (x - \beta y)(x - \gamma y) = 0$$

are found by means of the Jacobian of these two functions, viz. of the quadrics

$$(2, \ -\delta - \alpha, \ \ 2\delta\alpha \between x, \ y)^2,$$

$$(2, \ -\beta - \gamma, \ \ 2\beta\gamma \between x, \ y)^2,$$

which is

$$(\delta + \alpha - \beta - \gamma, \quad -\delta\alpha + \beta\gamma, \quad \delta\alpha(\beta + \gamma) - \beta\gamma(\delta + \alpha) \between x, \ y)^2,$$

viz. a quadratic factor of the cubicovariant; and forming the other two factors, there is no difficulty in seeing that any one of these is the Jacobian of the other two.

145. In the case of a pair of equal roots, we have

$$a^{-1} U = \ (x - \alpha y)^2 (x - \gamma y)(x - \delta y),$$

$$a^{-2} I = \ \tfrac{1}{12} \ (\alpha - \gamma)^2 (\alpha - \delta)^2,$$

$$a^{-3} J = -\tfrac{1}{216} (\alpha - \gamma)^3 (\alpha - \delta)^3,$$

$$\square \ = \ 0,$$

$$a^{-2} H = -\tfrac{1}{48} \{2(\alpha - \gamma)^2 (x - \delta y)^2 + 2(\alpha - \delta)^2 (x - \gamma y)^2 + (\gamma - \delta)^2 (x - \alpha y)^2\} (x - \alpha y)^2,$$

$$a^{-3} \Phi = \ \tfrac{1}{32} (\gamma - \delta)^2 (2\alpha - \gamma - \delta, \ \gamma\delta - \alpha^2, \ \gamma\alpha^2 + \delta\alpha^2 - 2\gamma\alpha\delta \between x, \ y)^2 (x - \alpha y)^4.$$

In the case of two pairs of equal roots, we have

$$\begin{aligned}
a^{-1}U &= (x-\alpha y)^2(x-\gamma y)^2,\\
a^{-2}I &= \tfrac{1}{12}(\alpha-\gamma)^4,\\
a^{-3}J &= -\tfrac{1}{216}(\alpha-\gamma)^6,\\
\square &= 0,\\
a^{-2}H &= -\tfrac{1}{12}(\alpha-\gamma)^2(x-\alpha y)^2(x-\gamma y)^2,\\
\Phi &= 0;
\end{aligned}$$

these values give also

$$6IH-9JU=0.$$

146. In the case of three equal roots, we have

$$\begin{aligned}
a^{-1}U &= (x-\alpha y)^3(x-\delta y),\\
I &= 0,\quad J=0,\quad \square=0,\\
a^{-2}H &= -\tfrac{1}{48}(\alpha-\delta)^2\{2(x-\delta y)^2+(x-\alpha y)^2\}(x-\alpha y)^2,\\
a^{-3}\Phi &= \tfrac{1}{32}(\alpha-\delta)^3(x-\alpha y)^6;
\end{aligned}$$

and in the case of four equal roots, we have

$$\begin{aligned}
a^{-1}U &= (x-\alpha y)^4,\\
I &= 0,\quad J=0,\quad \square=0,\\
H &= 0,\quad \Phi=0.
\end{aligned}$$

The preceding formulæ, for the case of equal roots, agree with the results obtained in my memoir on the conditions for the existence of given systems of equalities between the roots of an equation.

Addition, 7th October, 1858.

Covariant and other Tables (binary quadrics Nos. 25 bis, 29 A, 49 A, and 50 bis).

Mr Salmon has pointed out to me, that in the Table No. 25 of the simplest octinvariant of a binary quintic[1], the coefficients -210, -17, $+18$ and $+38$ are erroneous, and has communicated to me the corrected values, which I have since verified: the terms, with the corrected values of the coefficients, are [shown in the Table]

No. 25 bis.

[The terms with the erroneous coefficients were abc^2d^2ef, ac^5f^2, $b^4d^2f^2$, bc^3d^3e; the correct values -220, -27, $+22$, and $+74$ of the coefficients are given in the Table Q, No. 25, p. 288.]

[1] Second Memoir, *Philosophical Transactions*, t. CXLVI. (1856) p. 125.

Mr Salmon has also performed the laborious calculation of Hermites' 18-thic invariant of a binary quintic, and has kindly permitted me to publish the result, which is given in the following Table:

No. 29 A.

[This is the Table W No. 29 A given pp. 299—303, the form being slightly altered as appears p. 282.]

Mr Salmon has also remarked to me, that in the Table No. 50 of the cubinvariant of a binary dodecadic[1], the coefficients are altogether erroneous. There was, in fact, a fundamental error in the original calculation; instead of repeating it, I have, with a view to the deduction therefrom of the cubinvariant (see Fourth Memoir, No. 78), first calculated the dodecadic quadricovariant, the value of which is given in the following Table:

No. 49 A.

[For this Table see p. 319.]

It is now very easy to obtain the cubinvariant, which is

No. 50 bis.

[This is the Table No. 50, p. 319, the original No. 50 with coefficients which were altogether erroneous having been omitted.]

[1] Third Memoir, *Philosophical Transactions*, t. CXLVI. (1856) p. 635.