54.

ON CERTAIN INEQUALITIES RELATING TO PRIME NUMBERS.

[*Nature*, XXXVIII. (1888), pp. 259—262.]

I SHALL begin with a method of proving that the number of prime numbers is infinite, which is not new, but which it is worth while to recall as an introduction to a similar method, by series, which will subsequently be employed in order to prove that the number of primes of the form $4n+3$, as also of the form $6n+5$, is infinite.

It is obvious that the reciprocal of the product

$$\left(1-\frac{1}{p_1}\right)\left(1-\frac{1}{p_2}\right)\left(1-\frac{1}{p_3}\right)\dots\left(1-\frac{1}{p_{N.p}}\right)$$

(where p_i means the ith in the natural succession of primes, and $p_{N.p}$ means the highest prime number not exceeding N)* will be equal to

$$\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\dots+\frac{1}{N}+R,$$

and therefore greater than $\log N$ (R consisting exclusively of positive terms).

Hence $$\left(1+\frac{1}{p_1}\right)\left(1+\frac{1}{p_2}\right)\dots\left(1+\frac{1}{p_{N.p}}\right)>M\log N,$$

where $$M=\left(1-\frac{1}{p_1{}^2}\right)\left(1-\frac{1}{p_2{}^2}\right)\dots\left(1-\frac{1}{p_{N.p}{}^2}\right),$$

and is therefore greater than $\dfrac{2}{\pi}$.

Hence the number of terms in the product must increase indefinitely with N.

By taking the logarithms of both sides we obtain the inequality

$$S_1-\frac{1}{2}S_2+\frac{1}{3}S_3-\frac{1}{4}S_4+\dots>\log\log N+\log M,$$

* $N.p$ itself of course denotes in the above notation the number of primes (p) not exceeding N.

where in general S_i means the sum of inverse ith powers of all the primes not exceeding N; and accordingly is finite, except when $i=1$, for any value of N. We have therefore

$$S_1 > \log\log N + \text{Const.}$$

The actual value of S_1 is observed to differ only by a limited quantity from the second logarithm of N, but I am not aware whether this has ever been strictly proved.

Legendre has found that for large values of N

$$\left(1-\frac{1}{3}\right)\left(1-\frac{1}{5}\right)\dots\left(1-\frac{1}{p_{N.p}}\right)=\frac{1{\cdot}104}{\log N}.$$

Consequently

$$\left(1-\frac{1}{p_1}\right)\left(1-\frac{1}{p_2}\right)\dots\left(1-\frac{1}{p_{N.p}}\right)=\frac{{\cdot}552}{\log N}.$$

This would show that the value of our R bears a finite ratio to $\log N$; calling it $\theta \log N$ we obtain, according to Legendre's formula,

$$\frac{1}{1+\theta}={\cdot}552, \text{ which gives } \theta={\cdot}811,$$

so that the nebulous matter, so to say, in the expansion of the reciprocal of the product of the differences between unity and the reciprocals of all the primes not exceeding a given number, stands in the relation of about 4 to 5 to the condensed portion consisting of the reciprocals of the natural numbers.

I will now proceed to establish similar inequalities relating to prime numbers of the respective forms $4n+3$ and $6n+5$.

Beginning with the case $4n+3$, I shall use q_j to signify the jth in the natural succession of primes of the form $4n+3$, and $q_{N.q}$ to signify the highest q not exceeding N, $N.q$ itself signifying the number of q's not exceeding N.

Let us first, without any reference to convergence, consider the product obtained by the usual mode of multiplication of the infinite series

$$S=1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\dots \textit{ad inf.}$$

by the product

$$\frac{1}{1-\frac{1}{2}}\cdot\frac{1+\frac{1}{q_1}}{1-\frac{1}{q_1}}\cdot\frac{1+\frac{1}{q_2}}{1-\frac{1}{q_2}}\cdot\frac{1+\frac{1}{q_3}}{1-\frac{1}{q_3}}\dots \textit{ad inf.}$$

It is clear that the effect of the multiplication of S by the numerator of the above product will be to deprive the series S of all its negative terms. Then the effect of dividing by the denominator of the product, with the exception of the factor $1-\frac{1}{2}$, will be to restore all the obliterated terms, but with the sign + instead of −. Lastly, the effect of multiplying by the reciprocal of $(1-\frac{1}{2})$ will be to supply the even numbers that were wanting in the denominators of the terms of S, and we shall thus get the indefinite series

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\dots \textit{ad inf.}$$

Call now

$$Q_N=\frac{1}{1-\frac{1}{2}}\cdot\frac{1+\frac{1}{q_1}}{1-\frac{1}{q_1}}\cdot\frac{1+\frac{1}{q_2}}{1-\frac{1}{q_2}}\cdots\frac{1+\frac{1}{q_{N.q}}}{1-\frac{1}{q_{N.q}}}.$$

Q_N, which is finite when N is finite, may be expanded into an infinite aggregate of positive terms, found by multiplying together the series

$$1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\dots$$

$$1+\frac{2}{q_1}+\frac{2}{q_1^2}+\frac{2}{q_1^3}+\dots$$

$$1+\frac{2}{q_2}+\frac{2}{q_2^2}+\frac{2}{q_2^3}+\dots$$

....

....

$$1+\frac{2}{q_{N.q}}+\frac{2}{q_{N.q}^2}+\frac{2}{q_{N.q}^3}+\dots.$$

Let $$S_N=1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\frac{1}{11}+\dots\pm\frac{1}{N},$$

then from what has been said it is obvious that we may write

$$Q_NS_N=1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\dots+\frac{1}{N}+V-R,$$

where V and R may be constructed according to the following rule: Let the denominator of any term in the aggregate Q_N be called t, and let θ be the smallest odd number which, multiplied by t, makes $t\theta$ greater than N; then if θ is of the form $4n+1$ it will contribute to V a portion represented by the product of the term by some portion of the series S_N of the form

$$\frac{1}{\theta}-\frac{1}{\theta+2}+\frac{1}{\theta+4}-\dots$$

and if θ is of the form $4n+3$ it will contribute to $-R$ a portion equal to the term multiplied by a series of the form

$$-\frac{1}{\theta}+\frac{1}{\theta+2}-\frac{1}{\theta+4}+\dots.$$

Hence R is made up of the sum of products of portions of the aggregate Q_N multiplied respectively by the series

$$\frac{1}{3}-\frac{1}{5}+\frac{1}{7}-\frac{1}{9}+\frac{1}{11}-\frac{1}{13}+\dots$$

$$\frac{1}{7}-\frac{1}{9}+\frac{1}{11}-\frac{1}{13}+\dots$$

$$\frac{1}{11}-\frac{1}{13}+\dots$$

of which the greatest is obviously the first, whose value is $1-S_N$.

Consequently R must be less than the total aggregate Q_N multiplied by $1-S_N$.

Therefore

$$Q_N S_N + Q_N(1-S_N) > 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\dots+\frac{1}{N} > \log N,$$

that is, $$Q_N > \log N,$$

from which it follows that when N increases indefinitely the number of factors in Q_N also increases indefinitely, and there must therefore be an infinite number of primes of the form $4n+3$.

Denoting by M_N the quantity

$$\left(1-\frac{1}{q_1{}^2}\right)\left(1-\frac{1}{q_2{}^2}\right)\dots\left(1-\frac{1}{q_{N.q}{}^2}\right)$$

we obtain the inequality

$$\left(1+\frac{1}{q_1}\right)^2\left(1+\frac{1}{q_2}\right)^2\dots\left(1+\frac{1}{q_{N.q}}\right)^2 > \frac{1}{2}M_N\log N,$$

and taking the logarithms of both sides

$$\Sigma_1-\frac{1}{2}\Sigma_2+\frac{1}{3}\Sigma_3-\dots > \frac{1}{2}\log\log N+\frac{1}{2}\log M_N-\frac{1}{2}\log 2,$$

where in general Σ_i denotes the sum of the ith powers of the reciprocals of all prime numbers of the form $4n+3$ not surpassing N.

Hence it follows that $\Sigma_1 > \frac{1}{2}\log\log N$.

If we could determine the ultimate ratio of the sum of those terms of Q_N whose denominators are greater than N to the total aggregate, and should find that μ, the limiting value of this ratio, is not unity, then the method employed to find an inferior limit would enable us also to find a superior limit to Q_N; for we should have $V < \mu Q_N$ added to the sum of portions of what remains of the aggregate when μQ_N is taken from it multiplied respectively by the several series

$$\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\frac{1}{11}+\frac{1}{13}-\frac{1}{15}+\dots \textit{ ad inf.}$$

$$\frac{1}{9}-\frac{1}{11}+\frac{1}{13}-\frac{1}{15}+\dots \textit{ ad inf.}$$

$$\frac{1}{13}-\frac{1}{15}+\dots \textit{ ad inf.}$$

the total value of the sum of which products would evidently be less than

$$(1-\mu)\left(S-1+\frac{1}{3}\right)Q_N.$$

Hence the total value of V would be less than

$$\mu Q_N S+(1-\mu)\,Q_N\left(S-\frac{2}{3}\right),$$

that is, less than $$Q_N S-\frac{2}{3}(1-\mu)\,Q_N,$$

and consequently we should have

$$\frac{2}{3}(1-\mu)\,Q_N<\log N,$$

that is $$Q_N<\frac{3}{2(1-\mu)}\log N.$$

From which we may draw the important conclusion that if μ is less than 1, that is, if when N is infinite the portion of the aggregate $S_N Q_N$ comprising the terms whose denominators exceed N does not become infinitely greater than the remaining portion, the sum of the reciprocals of all the prime numbers of the form $4n+3$ not exceeding N would differ by a limited quantity from half the second logarithm of N.

A precisely similar treatment may be applied to prime numbers of the form $6n+5$. We begin with making

$$S_N=1-\frac{1}{5}+\frac{1}{7}-\frac{1}{11}+\frac{1}{13}-\frac{1}{17}+\frac{1}{19}-\dots.$$

We write

$$Q_N=\frac{1}{1-\frac{1}{2}}\cdot\frac{1}{1-\frac{1}{3}}\cdot\frac{1+\frac{1}{r_1}}{1-\frac{1}{r_1}}\cdot\frac{1+\frac{1}{r_2}}{1-\frac{1}{r_2}}\cdots\frac{1+\frac{1}{r_{N.r}}}{1-\frac{1}{r_{N.r}}}.$$

We make $$Q_N S_N=1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\dots+\frac{1}{N}+V-R.$$

We prove as before that $$R<(1-S)\,Q_N,$$

and thus obtain $$Q_N>\log N,$$

and then putting $$M_N = \left(1 - \frac{1}{r_1^2}\right)\left(1 - \frac{1}{r_2^2}\right) \dots \left(1 - \frac{1}{r_{N.r}^2}\right),$$

and finally noticing that $$\frac{1}{1-\frac{1}{2}} \cdot \frac{1}{1-\frac{1}{3}} = 3,$$

we obtain $$\left(1 + \frac{1}{r_1}\right)^2\left(1 + \frac{1}{r_2}\right)^2 \dots \left(1 + \frac{1}{r_{N.r}}\right)^2 > \frac{1}{3} M_N \log N.$$

Taking the logarithms of both sides of the equation, we find

$$\Theta_1 - \frac{1}{2}\Theta_2 + \frac{1}{3}\Theta_3 - \dots > \frac{1}{2}\log\log N + \frac{1}{2}\log M_N - \frac{1}{2}\log 3,$$

where Θ_i means the sum of ith powers of the reciprocals of all the prime numbers, not exceeding N, of the form $6n+5$.

Either from this equation or from the one from which it is derived it at once follows that the number of primes of the form $6n+5$ is greater than any assignable limit.

Parallel to what has been shown in the preceding case, if it could be ascertained that the sum of the terms of the aggregate Q_N whose denominators do not exceed N bears a ratio which becomes indefinitely small to the total aggregate, it would follow by strict demonstration that the sum of the reciprocals of the primes of the form $6n+5$ inferior to N would always differ by a limited quantity from the half of the second logarithm of N.

It is perhaps worthy of remark that the infinitude of primes of the forms $4n+3$ and $6n+5$ may be regarded as a simple rider to Euclid's proof (Book IX., Prop. 20) of the infinitude of the number of primes in general.

The point of this is somewhat blunted in the way in which it is presented in our ordinary text-books on arithmetic and algebra.

What Euclid gives is something more than this*: his statement is, "There are more prime numbers than any proposed multitude (πλῆθος) of prime numbers"; which he establishes by giving a formula for finding at least one more than any proposed number. He does not say, as our text-book writers do, "if possible let $A, B, \dots C$ be *all* the prime numbers," &c., but simply that if $A, B, \dots C$ are *any* proposed prime numbers, one or more additional ones may be found by adding unity to their product which will either itself be a prime number, or contain at least one additional prime; which is all that can correctly be said, inasmuch as the augmented product may be the power of a prime.

* Whereas the English elementary book writers content themselves with showing that to suppose the number of primes finite involves an absurdity, Euclid shows how from any given prime or primes to generate an infinite succession of primes.

Thus from one prime number arbitrarily chosen, a progression may be instituted in which one new prime number at least is gained at each step, and so an indefinite number may be found by Euclid's formula: for example, 17 gives birth to 2 and 3; 2, 3, 17 to 103; 2, 3, 17, 103 to 7, 19, 79; and so on.

We may vary Euclid's mode of generation and avoid the transcendental process of decomposing a number into its prime factors by using the more general formula, $a, b, \ldots c+1$, where $a, b, \ldots c$, are any numbers relatively prime to each other; for this formula will obviously be a prime number or contain one or more distinct factors relatively prime to $a, b, \ldots c$.

The effect of this process will be to generate a continued series of numbers all of which remain prime to each other: if we form the progression

$$a,\ a+1,\ a^2+a+1,\ a(a+1)(a^2+a+1)+1, \ldots$$

and call these successive numbers

$$u_1,\ u_2,\ u_3,\ u_4,\ \ldots$$

we shall obviously have $\qquad u_{x+1} = u_x{}^2 - u_x + 1.$

It follows at once from Euclid's point of view that no primes contained in any term up to u_x can appear in u_{x+1}, so that all the terms must be relatively prime to each other. The same consequence follows *a posteriori* from the scale of relation above given; for, as I had occasion to observe in the *Comptes Rendus* for April 1888 [see p. 620, below], if dealing only with rational integer polynomials,

$$\phi(x) = (x-a) f(x) + a,$$

then, whatever value, c, we give to x, no two forms $\phi^i(c)$, $\phi^j(c)$ can have any common measure not contained in a: in this case $\phi(x) = (x-1)x+1$; so that $\phi^i(c)$ and $\phi^j(c)$ must be relative primes for all values of i and j*.

It is worthy of remark that all the primes, other than 3, implicitly obtained by this process will be of the form $6i+1$.

Euclid's own process, or the modified and less transcendental one, may be applied in like manner to obtain a continual succession of primes of the form $4n+3$ and $6n+5$.

As regards the former, we may use the formula

$$2 \,.\, a \,.\, b \ldots c + 1$$

(where $a, b, \ldots c$ are any "proposed" primes of the form $4n+3$), which will necessarily be of the form $4n+3$, and must therefore contain *one* factor at least of that form.

* Another theorem of a similar kind is that, whatever integer polynomial $\phi(x)$ may be, if i, j have for their greatest common measure k, then $\phi^k[\phi(0)]$ will be the greatest common measure of $\phi^i[\phi(0)]$, $\phi^j[\phi(0)]$.

As regards the latter, we may employ the formula

$$3 \,.\, a \,.\, b \ldots c + 2$$

(where $a, b, \ldots c$ are each of the form $6n+5$), which will necessarily itself be, and therefore contain *one* factor at least, of that form.

The scale of relation in the first of these cases will be, as before,

$$u_{x+1} = u_x^2 - u_x + 1\,;$$

so that each term in the progression, abstracting 3, will be of the form $4i+3$ and $6j+1$ conjointly, and consequently of the form $12n+7$; as for example,

$$3,\ 7,\ 43,\ 1807, \ldots.$$

In the latter case the scale of relation is

$$u_{x+1} = u_x^2 - 2u_x + 2,$$

which is of the form $(u_x - 2)\,u_x + 2$. It is obvious that in each progression at each step one new prime will be generated, and thus the number of ascertained primes of the given form go on indefinitely increasing, as also might be deduced *a posteriori* by aid of the general formula above referred to from the scale of relation applicable to each. Each term in the second case (the term 3, if it appears, excepted) will be simultaneously of the form $6i-1$ and $4j+1$, and consequently of the form $12n+5$, as in the example 5, 17, 257, 65537, ...

The same simple considerations *cease* to apply to the genesis of primes of the forms $4n+1$, $6n+1$. We may indeed apply to them the formulae

$$(2 \,.\, a \,.\, b \ldots c)^2 + 1 \quad \text{and} \quad 3\,(a \,.\, b \ldots c)^2 + 1$$

respectively, but then we have to draw upon the theory of quadratic forms in order to learn that their divisors are of the form $4n+1$ and $6n+1$ respectively.

Of course the difference in their favour is that in their case *all* the divisors locked up in the successive terms of the two progressions respectively are of the prescribed form; whereas in the other two progressions, whose theory admits of so much simpler treatment, we can only be assured of the presence of *one* such factor in each of the several terms.

Euler has given the values of two infinite products, without any evidence of their truth except such as according to the lax method of dealing with series without regard to the laws of convergence prevalent in his day, and still held in honour in Cambridge down to the times of Peacock, De Morgan, and Herschel inclusive (and this long after Abel had justly denounced the use of divergent series as a crime against reason), was erroneously supposed to amount to a proof, from which the same consequences may be derived as shown in the foregoing pages, and something more besides*. These two theorems are

$$(1)\qquad \frac{3}{3+1}\cdot\frac{5}{5-1}\cdot\frac{7}{7+1}\cdot\frac{11}{11+1}\cdot\frac{13}{13-1}\cdots=\frac{\pi}{4}$$

(where, corresponding to the primes 3, 7, 11, ... of the form $4n+3$, the factors of the product on the left are

$$\frac{3}{3+1},\ \frac{7}{7+1},\ \frac{11}{11+1},\cdots$$

all of them with the sign + in the denominator; while the fractions corresponding to primes of the form $4n+1$ have the − sign in their denominators).

$$(2)\qquad \frac{5}{5+1}\cdot\frac{7}{7-1}\cdot\frac{11}{11+1}\cdot\frac{13}{13-1}\cdot\frac{17}{17+1}\cdots=\frac{\pi}{2}\sqrt{3}$$

where, as in the previous product, the sign in the denominator of each fraction depends on the form of the prime to which it corresponds (being + for primes of the form $6n-1$, and − for primes of the form $6n+1$).

Dr J. P. Gram (*Mémoires de l'Académie Royale de Copenhague*, 6me série, Vol. II. p. 191) refers to a paper by Mertens ("Ein Beitrag zur analytischen Zahlentheorie," *Borchardt's Journal*, Bd 78), as one in which the truth of the first of the two theorems is demonstrated—"fuldstœndigt Bevis af Mertens" are Gram's words†.

* It follows from the first of these theorems that with the understanding that no denominator is to exceed n (an indefinitely great number),

$$\left(1+\frac{1}{3}\right)\left(1+\frac{1}{7}\right)\left(1+\frac{1}{11}\right)\left(1+\frac{1}{19}\right)\cdots$$

bears a finite ratio to

$$\left(1+\frac{1}{5}\right)\left(1+\frac{1}{13}\right)\left(1+\frac{1}{17}\right)\cdots$$

so that as their *product* is known to be infinite, each of these two partial products must be separately infinite; in like manner from Euler's second theorem a similar conclusion may be inferred in regard to each of the two products

$$\left(1+\frac{1}{5}\right)\left(1+\frac{1}{11}\right)\left(1+\frac{1}{17}\right)\left(1+\frac{1}{23}\right)\left(1+\frac{1}{29}\right)\left(1+\frac{1}{41}\right)\cdots$$

and

$$\left(1+\frac{1}{7}\right)\left(1+\frac{1}{13}\right)\left(1+\frac{1}{19}\right)\left(1+\frac{1}{31}\right)\cdots.$$

† It always seems to me absurd to speak of a complete proof, or of a theorem being rigorously demonstrated. An incomplete proof is no proof, and a mathematical truth not rigorously demonstrated is not demonstrated at all. I do not mean to deny that there are mathematical truths, morally certain, which defy and will probably to the end of time continue to defy proof, as, for example, that every indecomposable integer polynomial function must represent an infinitude of primes. I have sometimes thought that the profound mystery which envelops our conceptions relative to prime numbers depends upon the limitation of our faculties in regard to time, which like space may be in its essence poly-dimensional, and that this and such sort of truths would become self-evident to a being whose mode of perception is according to *superficially* as distinguished from our own limitation to *linearly* extended time.

Assuming this to be the case, we shall easily find when N is indefinitely great, so that S_N becomes $\frac{\pi}{4}$,

$$Q_N S_N = \frac{1}{\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\dots\left(1-\frac{1}{N}\right)},$$

which, according to Legendre's empirical law (Legendre, *Théorie des Nombres*, 3rd edition, Vol. II. p. 67, Art. 397), is equal to $\frac{2\log N}{K}$, where $K = 1{\cdot}104$; and as we have written $Q_N S_N = \log N + (V - R)$, we may deduce, upon the above assumptions,

$$V - R = \left(\frac{2}{K} - 1\right)\log N = 0{\cdot}811 \dots \log N.$$

R, we know, is demonstrably less than $\left(1-\frac{\pi}{4}\right)\log N$, consequently V must be less than $(0{\cdot}812 + 0{\cdot}215)\log N$, that is, less than $1{\cdot}027 \log N$, and *a fortiori* the portion of the omnipositive aggregate Q_N, which consists of terms whose denominators exceed N, when N is indefinitely great, cannot be less than $\frac{4}{\pi}\left(1-\frac{\pi}{4}\right)\log N$, that is, $0{\cdot}273 \log N$.

Before concluding, let me add a word on Legendre's empirical formula for the value of

$$\left(1-\frac{1}{3}\right)\left(1-\frac{1}{5}\right)\dots\left(1-\frac{1}{p_{N.p}}\right),$$

referred to in the early part of this article.

If N is any odd number, the condition of its being a prime number is that when divided by any odd prime less than its own square root, it shall not leave a remainder zero. Now if N (an unknown odd number) is divided by p, its remainder is equally likely to be 0, 1, 2, 3, ... or $(p-1)$. Hence the chance that it is not divisible by p is $\left(1-\frac{1}{p}\right)$, and, if we were at liberty to regard the like thing happening or not for any two values of p within the stated limit as independent events, the expectation of N being a prime number would be represented by

$$\left(1-\frac{1}{3}\right)\left(1-\frac{1}{5}\right)\left(1-\frac{1}{7}\right)\left(1-\frac{1}{11}\right)\dots\left(1-\frac{1}{p_{N^{\frac{1}{2}}.p}}\right),$$

which, according to the formula referred to, for infinitely large values of N is equal to $\frac{1{\cdot}104}{\log N^{\frac{1}{2}}}$. It is rather more convenient to regard N as entirely unknown instead of being given as odd, on which supposition the chance of its being a prime would be $\frac{1{\cdot}104}{2\log N^{\frac{1}{2}}}$ or $\frac{1{\cdot}104}{\log N}$.

Hence for very large values of N the sum of the logarithms of all the primes inferior to N might be expected to be something like $(1{\cdot}104)N$. This does not contravene Tchebycheff's formula (Serret, *Cours d'Algèbre Supérieure*, 4me ed., Vol. II. p. 233), which gives for the limits of this sum AN and BN, where $A = 0{\cdot}921292$, and $B = \frac{6A}{5} = 1{\cdot}10555$; but does contravene the narrower limits given by my advance upon Tchebycheff's method [see Vol. III. of this Reprint, p. 530], according to which for A, B, we may write A_1, B_1, where

$$A_1 = 0{\cdot}921423,\ B_1 = 1{\cdot}076577\,*.$$

That the method of probabilities *may* sometimes be successfully applied to questions concerning prime numbers I have shown reason for believing in the two tables published by me [above, p. 101] in the *Philosophical Magazine* for 1883†.

* Namely $A_1 = \frac{51072}{50999}A$, and $B_1 = \frac{59595}{50999}A$, the values of which are incorrectly stated in the memoir. Strange to say, Dr Gram, in his prize essay, previously quoted, on the number of prime numbers under a given limit, has omitted all reference to this paper in his bibliographical summary of the subject, which is only to be accounted for by its having escaped his notice; a narrowing of the asymptotic limits assigned to the sum of the logarithms of the prime numbers series being the most notable fact in the history of the subject since the publication of Tchebycheff's memoir. Subjectively, this paper has a peculiar claim upon the regard of its author, for it was his meditation upon the two simultaneous difference-equations which occur in it that formed the starting-point, or incunabulum, of that new and boundless world of thought to which he has given the name of Universal Algebra. But, apart from this, that the superior limit given by Tchebycheff as 1·1055 should be brought down by a more stringent solution of his own inequalities to only 1·076577—in other words, that the excess above the probable mean value (unity) should be reduced to little more than $\frac{2}{3}$rds of its original amount—is in itself a surprising fact. Perhaps the numerous (or innumerable) misprints and arithmetical miscalculations which disfigure the paper may help to account for the singular neglect which it has experienced. It will be noticed that the mean of the limits of Tchebycheff is 1·01342, the mean of the new limits being 0·99900. The excess in the one case above and the defect in the other below the probable true mean are respectively 0·01342 and 0·00100.

† A principle precisely similar to that employed above if applied to determining the number of reduced proper fractions whose denominators do not exceed a given number n, leads to a correct result. The expectation of two numbers being prime to each other will be the product of the expectations of their not being each divisible by any the same prime number. But the probability of one of them being divisible by i is $\frac{1}{i}$, and therefore of two of them being not each divisible by i is $\frac{1}{i^2}$. Hence the probability of their having no common factor is

$$\left(1-\frac{1}{4}\right)\left(1-\frac{1}{9}\right)\left(1-\frac{1}{49}\right)\left(1-\frac{1}{121}\right)\ldots\ ad\ inf.,\ \text{that is, is } \frac{6}{\pi^2}.$$

If, then, we take two sets of numbers, each limited to n, the probable number of relatively prime combinations of each of one set with each of the other should be $\frac{6n^2}{\pi^2}$, and the number of reduced proper fractions whose denominators do not exceed n should be the half of this or $\frac{3n^2}{\pi^2}$. I believe M. Césaro has claimed the prior publication of this mode of reasoning, to which he is heartily welcome. The number of these fractions is the same thing as the sum of the *totients* of all numbers not exceeding n. In the *Philosophical Magazine* for 1883 (Vol. xv. p. 251), a table of these sums of totients has been published by me for all values of n not exceeding 500, and [above, p. 101] in the same year (Vol. xvi. p. 231) the table was extended to values of n not exceeding 1000. In every case without any exception the estimated value of this totient sum is found to be intermediate between

$$\frac{3n^2}{\pi^2} \text{ and } \frac{3(n+1)^2}{\pi^2}.$$

Calling the totient sum to n, $T(n)$, I stated the exact equation

$$T(n)+T\left(\frac{n}{2}\right)+T\left(\frac{n}{3}\right)+T\left(\frac{n}{4}\right)+\ldots=\frac{n^2+n}{2},$$

from which it is capable of proof, without making any assumption as to the form of Tn, that its asymptotic value is $\frac{3n^2}{\pi^2}$. The functional equation itself is merely an integration (so to say) of the well-known theorem that any number is equal to the sum of the totients of its several divisors. The introduction to these tables will be found very suggestive, and besides contains an interesting bibliography of the subject of Farey series (*suites de Farey*), comprising, among other writers upon it, the names of Cauchy, Glaisher, and Sir G. Airy, the last-named as author of a paper on toothed wheels, published, I believe, in the "Selected Papers" of the Institute of Mechanical Engineers. The last word on the subject, as far as I am aware, forms one of the *interludes*, or rather the *postscript*, to my "Constructive Theory of Partitions," published in the *American Journal of Mathematics* [above, p. 55].