68.

ON AN ARITHMETICAL THEOREM IN PERIODIC CONTINUED FRACTIONS.

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THE well-known form of continued fraction for the square root of N, an integer, is

$$(a\,;\ b,\ c,\ d,\ \dots,\ d,\ c,\ b,\ 2a\,;\ b,\ c,\ d,\ \dots,\ d,\ c,\ b,\ 2a\,;\ \text{indefinitely continued})$$

which, if we denote the type $a, b, c, d, \dots, d, c, b, a$ by t, may be written under the more convenient form

$$(t,\ 0,\ t,\ 0,\ t,\ 0, \dots \textit{ ad inf.}).$$

If now we use $[t]$ to signify the cumulant of which t is the type, and $[`t]$, $[t']$, $[`t']$ respectively, the cumulants of the types got by cutting off a from either end and from both ends of t, it is easily shown that *whatever* numbers $a, b, c, \dots$ represent, the value of the continued fraction $\{(t, 0)^\infty\}$ is $\sqrt{\dfrac{[t]}{[`t']}}$, so that if $\{(t, 0)^\infty\}$ represents the square root of an integer, $[t]$ must be divisible by $[`t']$.

At first sight one would imagine that it would be a difficult matter to give a rule for determining whether such condition is fulfilled or not by any assigned value of the symmetrical type t, but Mr C. E. Bickmore, of New College, Oxford, has noticed that the case is quite otherwise, for that if we put t under the form a, τ, a, then, in order that $\{(a, \tau, a, 0)^\infty\}$ may satisfy the requirement of being the square root of an integer, the sufficient and necessary condition is the equivalence

$$2a \equiv (-)^\mu [\tau'] [`\tau'] \,(\text{mod. } [\tau]),$$

where μ is the number of elements in τ.

Consequently τ may be taken quite arbitrarily, and then an infinite number of values be assigned to a, except in the case where $[\tau]$ is even, and at the same time $[\tau']$ and $[`\tau']$ are each of them odd.

The proof in my notation is as follows:

Since $t = a, \tau, a$, we have $`t' = \tau$, and consequently $\frac{[t]}{[`t']}$ will be an integer if

$$[a, \tau, a] \equiv 0 \ (\text{mod. } [\tau]).$$

Expanding and remembering that $[`\tau] = [\tau']$ (the type τ being symmetrical), we obtain

$$a^2 [\tau] + 2a [\tau'] + [`\tau'] \equiv 0 \ (\text{mod. } [\tau]).$$

Hence $$2a [\tau'] + [`\tau'] \equiv 0 \ (\text{mod. } [\tau]), \tag{1}$$

and $$2a [\tau']^2 + [\tau'] [`\tau'] \equiv 0 \ (\text{mod. } [\tau]). \tag{2}$$

But $$[\tau']^2 - [\tau] [`\tau'] = (-1)^{\mu+1},$$

so that $$[\tau']^2 \equiv (-1)^{\mu+1} \ (\text{mod. } [\tau]),$$

and therefore (2) becomes

$$2a \equiv (-)^{\mu} [\tau'] [`\tau'] \ (\text{mod. } [\tau]), \tag{3}$$

which is thus shown to be a necessary condition.

It is also a sufficient condition, for multiplying (3) by $[\tau']$ we have

$$2a [\tau'] \equiv (-)^{\mu} [\tau']^2 [`\tau'] \ (\text{mod. } [\tau]),$$

or, since $$[\tau']^2 \equiv (-)^{\mu+1} \ (\text{mod. } [\tau]),$$

$$2a [\tau'] \equiv - [`\tau'] \ (\text{mod. } [\tau]),$$

which is the same as (1).

Suppose now that $`\tau'$ is given and that we wish to ascertain if $a$ can be found of such a value that the congruence (3) shall be soluble. This will obviously be the case if $[\tau]$ is odd. It will also be the case if $[\tau]$ is even, provided $[`\tau']$ is also even, and only in that case; for, when $[\tau]$ is even, then by virtue of the equation

$$[`\tau'] [\tau] - [\tau']^2 = \pm 1,$$

$[\tau']$ must be odd.

We have, therefore, to find under what circumstances $[`\tau']$ will be odd and $[\tau]$ even; in all other cases but these the congruence (3) will be soluble, and then the most general value of $a$ will be any term in an arithmetical series of which the common difference is $[\tau]$, unless $[\tau]$ and $[`\tau']$ are both of them even, in which case the common difference will be $\frac{1}{2} [\tau]$.

I proceed now to give a rule for determining the possible and impossible cases of the solution of (3), to explain the grounds of which the following statement will suffice.

(1) The value of a cumulant is not affected by striking out any even number of consecutive zeros from its type.

(2) The *parity* (that is the character *quâ* the modulus 2) of any cumulant will not be affected if we strike out three consecutive odd terms, whether

they occur in the middle or at either extremity. For if t, τ be any two types, the cumulant

$$[t, 1, 1, 1, \tau] = 3[t][\tau] + 2[t'][\tau] + 2[t][`\tau] + [t'][`\tau]$$
$$\equiv [t][\tau] + [t'][`\tau] \text{ (mod. 2)},$$

that is $$\equiv [t, \tau] \text{ (mod. 2)}.$$

Also

$$[1, 1, 1, t] = [t, 1, 1, 1] = 3[t] + 2[`t] \equiv [t] \text{ (mod. 2)}.$$

(3) The value of any cumulant in the type of which 1, 0, 1 occurs anywhere is the same as if 2 is substituted for 1, 0, 1; and therefore its parity is not affected if the units on each side of the 0 are omitted.

In what precedes in Nos. (1), (2), (3) the result, to modulus 2, is obviously unaffected if for 0 we write any even and for 1 any odd number.

In order then to determine the parity of $[`\tau']$ and of $[\tau]$ we may proceed as follows:

Let τ be any assigned symmetrical type, $`\tau'$ will then represent the type divested of its two equal terminals.

Rules—(1) for each even number in $`\tau'$ write 0, and for each odd number, 1;

(2) elide any even number of consecutive *zeros*, and any number divisible by 3 of consecutive *units*;

(3) elide any pair of units lying on each side of a zero;

(4) repeat these processes as often as possible;

then, I say, eventually we must arrive at one or other of the six following irreducible types, namely

( ); 0; 1; 1, 1; 0, 1, 0; 0, 1, 1, 0*,

where ( ) means absolute vacuity; accordingly $`\tau'$ may be said to be affected with one or the other of these six *characters*.

If now the reduced form of $`\tau'$ is 0; 1, 1; 0, 1, 0, $[`\tau']$ is even, and the congruence (3) will be soluble. In the other three cases $[`\tau']$ is odd, but $[\tau]$ will also be odd unless its terminal elements are odd in the case where the reduced form of $`\tau'$ is ( ), and even for the reduced forms 1, and 0, 1, 1, 0.

In the following exhaustive table the second column indicates the evenness or oddness of the terminals of τ denoted by e and u respectively.

The third and fourth columns indicate the evenness or oddness (denoted as above) of $[`\tau']$ and $[\tau]$, along with the character of $`\tau'$ in the third column. In the fifth column the answer is given as to the determining congruence

* Except for the symmetrical form of τ there would be two additional (virtually undistinguishable) reduced forms 0, 1 and 1, 0.

being soluble or insoluble, denoted by s and i respectively; and the last column shows whether the common difference of the arithmetical series of the values of either terminal, in the case of solubility, is equal to the modulus $[\tau]$ or its moiety.

Cases	Terminals	$`\tau'$		$[\tau]$	Sol. or Insol.	C. D.
1	e	( )	u	u	s	$[\tau]$
2	u	( )	u	e	i	
3	e	1	u	e	i	
4	u	1	u	u	s	$[\tau]$
5	e	0, 1, 1, 0	u	e	i	
6	u	0, 1, 1, 0	u	u	s	$[\tau]$
7	e	0	e	e	s	$\frac{1}{2}[\tau]$
8	u	0	e	e	s	$\frac{1}{2}[\tau]$
9	e	1, 1	e	u	s	$[\tau]$
10	u	1, 1	e	u	s	$[\tau]$
11	e	0, 1, 0	e	u	s	$[\tau]$
12	u	0, 1, 0	e	u	s	$[\tau]$

The following examples are given to prevent the possibility of misapprehension in the application of the Algorithm.

(α) Let

$$\tau = 1, 9, 1, 1, 1, 2, 1, 7, 4, 2, 2, 2, 4, 7, 1, 2, 1, 1, 1, 9, 1.$$

$$\begin{aligned} \text{Then } `\tau' &\equiv 1, 1, 1, 1, 0, 1, 1, 0, 0, 0, 0, 0, 1, 1, 0, 1, 1, 1, 1 \\ &\equiv 0, \quad 1, \quad 0, 1, \quad 0 \\ &\equiv 0, \quad 0, \quad 0 \\ &\equiv 0. \end{aligned}$$

This corresponds to case (8), which is a soluble one, and accordingly we have from Degen's Table

$$\{(15, \tau, 15, 0)^{\infty}\} = \sqrt{(251)},$$

15 being the first term of an arithmetical series whose common difference is $\frac{1}{2}[\tau]$.

(β) Let $\quad \tau = 2, 3, 1, 2, 4, 1, 6, 6, 1, 4, 2, 1, 3, 2.$

$$\begin{aligned} \text{Then } `\tau' &\equiv 1, 1, 0, 0, 1, 0, 0, 1, 0, 0, 1, 1 \\ &\equiv 1, 1, \quad 1, \quad 1, \quad 1, 1 \\ &\equiv (\ ). \end{aligned}$$

This corresponds to the soluble case (1), and accordingly we find from Degen's Table $\{(10, \tau, 10, 0)^{\infty}\} = \sqrt{(109)}$; 10 being the first term of an arithmetical series whose common difference is $[\tau]$.