69.

ON A FUNICULAR SOLUTION OF BUFFON'S "PROBLEM OF THE NEEDLE" IN ITS MOST GENERAL FORM.

[*Acta Mathematica*, XIV. (1890–1), pp. 185—205.]

"...quaintly made of cords."
(*Two Gentlemen of Verona*, Act III. Sc. 1.)

THE founder of the theory of Local Probability appears to have been Buffon (better known as a Naturalist, but who began his career as a Mathematician). Among a few other questions of a similar kind, which he proposed in his *Essai d'Arithmétique Morale*, the one which has obtained the greatest notoriety is the celebrated one which goes by the name of the *Problème de l'Aiguille*, the purport of which is as follows.

On an area of indefinite extent (say a planked floor) a number of parallel straight lines are ruled at equal distances, upon which a needle, not long enough to cross more than one of the parallels at the same time, is thrown down: the probability is required of its falling in such a position as to be intersected by one of the parallels.

An easier question of the same kind, which Buffon treats before the other, is when a circle is used instead of the needle. This latter question he solves by simple geometrical considerations too obvious to need recapitulation; to obtain a solution of the former he, and after him Laplace, had recourse to a process of integration.

In a question given in the late Mr Todhunter's *Integral Calculus* (1st edition, 1857, p. 268) the solution of the problem is correctly stated for an ellipse, whose major axis is less than the distance between two consecutive parallels, instead of for a circle or straight line: this important step in the development of the theory is, I am informed, currently attributed to the late Mr Leslie Ellis, of the University of Cambridge.

In the year 1860, Lamé proposed to give a course of lectures on the subject at the Sorbonne, and, apparently without knowledge of the result contained in Todhunter's treatise, reproduced the solution for the ellipse and for any equilateral polygon. In the same year M. Emile Barbier, whose lamented decease occurred in the course of the present year and who had attended Lamé's lectures, discovered and published in *Liouville's Journal* for that year a universal solution for an undivided plane contour of any form whatever.

The subsequent history I am not able to trace further than to state that in Czuber's *Geometrische Wahrscheinlichkeiten* (Leipzig, 1884) Barbier's solution is extended to the case of any two rigidly connected convex figures (in a plane)*. I propose to give here the finishing stroke to the theory as regards plane figures by extending it to any number of them, rigidly connected and of any forms, in the same plane. It is always to be understood, in what precedes as in what follows, that the greatest diameter of the figure, or system of figures, is less than the distance between two consecutive parallels.

Barbier's principle (see Czuber, pp. 117, 125) leads at once to the conclusion that the probability of any figure (subject to the restriction above stated) intersecting the system of parallels is to certainty as the length of a cord stretched round the figure is to the circumference of a circle touched by two adjoining parallels†. This circumference (with a view to simplicity of expression) we shall adopt as the unit of length in all subsequent formulae.

By the disjunctive probability of a set of figures I shall understand the probability of *one or more* of them intersecting one of the parallels: by the conjunctive probability of the same, the probability of *all* of them intersecting one of the parallels.

I start from Barbier's theorem that for a single figure the probability of intersection is measured by the length of a stretched string passing round it: this, it should be observed, is universally true whether the contour be curvilinear or rectilinear or mixtilinear, composed of a single line straight or curved or of any number of such—a theorem almost unexampled for its generality. The disjunctive probability for any number of figures $A, B, C, \ldots, H$ I shall for the present denote by $A : B : C : \ldots : H$, the conjunctive by $A.B.C \ldots H$.

Let there be $n+1$ figures given, let p_i be the sum of the conjunctive and ϖ_i of the disjunctive probabilities for these figures taken i and i together; so that ϖ_1 and p_1 are identical, and ϖ_{n+1}, p_{n+1} are monomial quantities. Then by a universal theorem of *logic* we have the reciprocal formulae

$$\varpi_{n+1} = \sum_{i=1}^{i=n+1} (-)^{i+1} p_i, \qquad (1)$$

$$p_{n+1} = \sum_{i=1}^{i=n+1} (-)^{i+1} \varpi_i. \qquad (2)$$

* See *Postscriptum*, p. [679, below].

† The case of a straight line (the original question of the *needle*) may be made to fall under this rule: for the line, as Barbier has observed, may be regarded as an indefinitely narrow ellipse or other oval.

Let us now suppose that we have obtained expressions for the disjunctive and conjunctive probabilities of any number not exceeding n figures of any kind: we may extend these to the case of $n+1$ figures as follows.

(1) When the $n+1$ figures are so situated that it is impossible for all of them to be cut by the same straight line, we have $p_{n+1}=0$ so that ϖ_{n+1} can be found immediately in terms of $p_1, p_2, \ldots, p_n$ by using formula (1), or in terms of $\varpi_1, \varpi_2, \ldots, \varpi_n$ by using (2); that is ϖ_{n+1} can be found in terms of known quantities; for by hypothesis all the terms of p_i or of ϖ_i are known when i is any number not exceeding n.

(2) When all the $n+1$ figures are capable of being cut by the same straight line, let XY be some straight line which cuts them all and call the figures taken in the order in which they are cut by XY

$$A_1, A_2, A_3, \ldots, A_{n+1}\text{*}.$$

Let a stretched string be made to wind round these $n+1$ contours passing alternately from one side of XY to the other, as in Fig. 1, and crossing itself

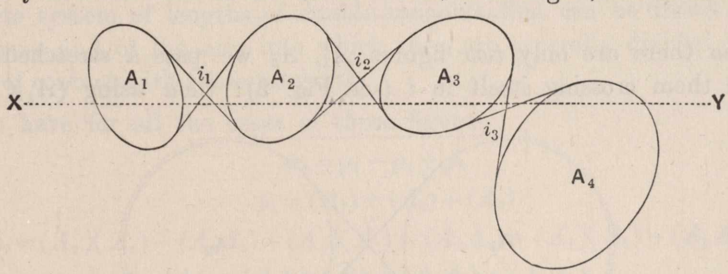

Fig. 1.

in the n points $i_1, i_2, \ldots, i_n$ lying between A_1, A_2; A_2, A_3; ... A_n, A_{n+1} respectively. Let us call the figures enclosed by the successive $n+1$ *loops* of the winding string

$$B_1, B_2, B_3, \ldots, B_{n+1}.$$

It is obvious that any straight line which cuts all these loops will cut all the given figures, and *vice versâ*.

Hence $$A_1 . A_2 . A_3 \ldots A_{n+1} = B_1 . B_2 . B_3 \ldots B_{n+1}.$$

Let P_i, Π_i represent what p_i, ϖ_i become when for the figures A we substitute the loops B, so that

$$\Pi_{n+1} = \sum_{i=1}^{i=n+1} (-)^{i+1} P_i,$$

$$P_{n+1} = \sum_{i=1}^{i=n+1} (-)^{i+1} \Pi_i,$$

and $$P_{n+1} = p_{n+1}.$$

* It may be well to draw at once attention to the fact that different systems of straight lines do not necessarily cut the figures $A_1, A_2, A_3, \ldots$ in the same order; as, for example, if three circles touch, or so nearly touch one another that each blocks the channel between the other two, straight lines may be drawn whose intersections with *any* one of the three shall be intermediate to their intersections with the other two.

Π_{n+1} is known by Barbier's rule, because the loops taken together form a single figure, in fact

$$\Pi_{n+1} = L,$$

where L is the length of the uncrossed string stretched round the system of figures B, which is no other than that stretched round the given figures A. Also, by hypothesis, Π_i is known for all values of i not exceeding n. We therefore know p_{n+1} which is the same as P_{n+1}. Hence ϖ_{n+1} is known from (1): thus then p_{n+1} and ϖ_{n+1} are both known, so that when the conjunctive and disjunctive probabilities are known in general for n figures they become known for $n+1$ figures; but when $n=1$, p_1 and ϖ_1 are equal to one another and to the length of a given stretched string. Hence, by the usual process of induction, we may conclude that the conjunctive and disjunctive probabilities for any number of figures can always be expressed as a linear function with positive and negative integer coefficients, or in a word as a Diophantine linear function, of a finite number of lengths of certain stretched strings.

When there are only *two* figures A_1, A_2 we pass a stretched string between them crossing itself in i (see Fig. 2): then using $(A_1 \times A_2)$ to

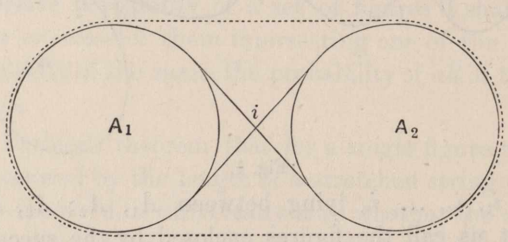

Fig. 2.

denote the length of this string, and (A_1A_2) to denote the length of the uncrossed string (indicated by dots in the figure) stretched round A_1, A_2 we have

$$\Pi_2 = (A_1 \times A_2) - P_2$$

and

$$\varpi_2 = (A_1) + (A_2) - p_2$$

(where (A_1), (A_2) denote the lengths of the separate bands round A_1, A_2 respectively).

But

$$\Pi_2 = (A_1A_2),$$

and consequently

$$p_2 = P_2 = (A_1 \times A_2) - (A_1A_2),$$

$$\varpi_2 = (A_1) + (A_2) + (A_1A_2) - (A_1 \times A_2).$$

We will now proceed to consider in detail the application of the inductive method to the case of *three* figures for which, since each of these may be replaced by a convex band passing round it, we may if we please for greater graphical simplicity substitute three convexes (that is contours which any secant must intersect in exactly two points). Many cases requiring separate discussion will arise, but one important consequence, rising to the dignity of a principle, which holds good whatever may be the number of figures, governs them all; namely that the final result for either probability is a linear homogeneous function of lengths of stretched bands drawn in various ways round the given figures and depending for their course on the forms and disposition of these figures exclusively, *wholly uninfluenced* by the presence of any points external to them. Lines drawn from the pointed ends, or apices, of the loops enclosing them do it is true make their appearance in the computations but, either coalesce into portions of the bands referred to, or else, entering in pairs with opposite algebraical signs, disappear from the final result. As a consequence, if for the sake of illustration we suppose the figures to be any closed *curves* without singular points, the probability, disjunctive or conjunctive, to be ascertained is a function exclusively of the complete system of lengths of double tangents that can be drawn between the curves and of the arcs into which they are severally divided by their points of contact with those tangents.

We have for all the cases of three figures

$$\varpi_3 = p_1 - p_2 + p_3$$

where $$p_1 = (A_1) + (A_2) + (A_3)$$

and $$p_2 = (A_2 \times A_3) - (A_2A_3) + (A_3 \times A_1) - (A_3A_1) + (A_1 \times A_2) - (A_1A_2).$$

Thus $$\varpi_3 - p_3 = (A_1) + (A_2) + (A_3) + (A_2A_3) + (A_3A_1) + (A_1A_2) - (A_2 \times A_3) - (A_3 \times A_1) - (A_1 \times A_2). \quad (3)$$

Similarly $$\Pi_3 - P_3 = (B_1) + (B_2) + (B_3) + (B_2B_3) + (B_3B_1) + (B_1B_2) - (B_2 \times B_3) - (B_3 \times B_1) - (B_1 \times B_2),$$

where B_1, B_2, B_3 are the loops of the string which passes round the figures A_1, A_2, A_3 and crosses itself at i and j, as shown in Fig. 3. But $P_3 = p_3$,

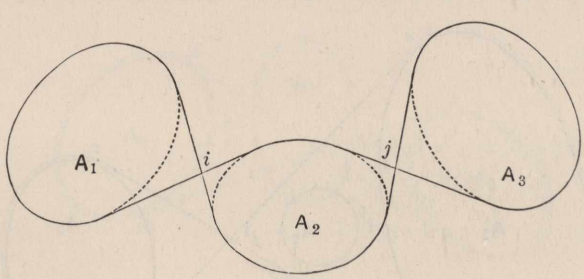

Fig. 3.

and Π_3 is the length of an uncrossed band stretched round the entire system of figures A_1, A_2, A_3 (which will be expressed in symbols by writing

$$\Pi_3 = (A_1A_2A_3).)$$

Hence $p_3 = (A_1A_2A_3) + (B_2 \times B_3) + (B_3 \times B_1)$
$+ (B_1 \times B_2) - (B_1) - (B_2) - (B_3) - (B_2B_3) - (B_3B_1) - (B_1B_2)$.

Moreover $(B_1 \times B_2) = (B_1) + (B_2)$

and $(B_2 \times B_3) = (B_2) + (B_3)$,

because B_1, B_2 and B_2, B_3 are pairs of consecutive loops. And whenever the three given figures are capable of being cut by a straight line in the order A_1, A_2, A_3 (that is except in the case $p_3 = 0$, which is separately considered)

$$(B_3B_1) = (A_3A_1),$$

because both the crossing points, i and j, of the looped string necessarily fall inside the uncrossed band round A_1, A_3. Thus the value of p_3 is given by the equation

$$p_3 = (A_1A_2A_3) - (A_1A_3) + (B_3 \times B_1) + (B_2) - (B_2B_3) - (B_1B_2) \quad (4)$$

which, for immediate purposes, we shall find convenient to write under the form

$$p_3 = (A_1A_2A_3) - (A_1A_3) + (B_2 \times B_3) - (B_2B_3) + (B_3 \times B_1) - (B_1B_2) - (B_3). \quad (5)$$

We shall apply the formula to the two classes which between them comprise all the cases of three figures, namely

Class A. One of the figures, which we call A_2, lies either wholly or partially inside the crossed band round the other two.

Class B. Each figure lies entirely outside the crossed band round the other two.

In Class A we recognize three species, namely

Aa. The figure A_2 does not cut either of the crossed strings ab, cd of the band looped round A_1, A_3 (Fig. 4), but lies wholly in the same loop as one of them, which we call A_1.

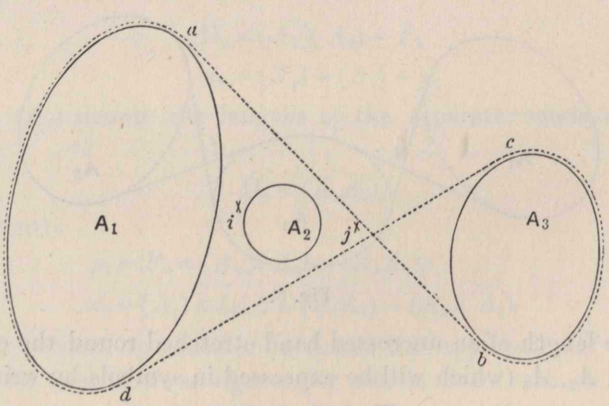

Fig. 4.

Ab. The figure A_2 cuts one, but not both, of the crossed strings *ab*, *cd* (Fig. 5), and part of it lies in the same loop as A_1.

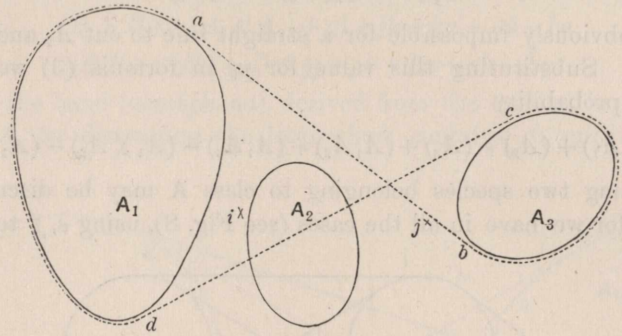

Fig. 5.

Ac. The figure A_2 cuts both the crossed strings *ab*, *cd* (Figs. 6 and 7) and part of it lies in the same loop as A_1.

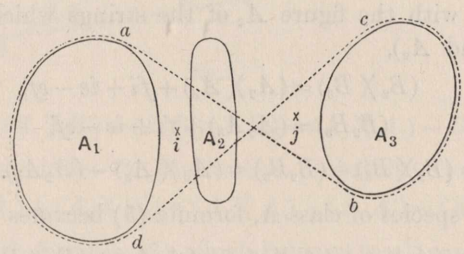

Fig. 6.

To avoid complicating these figures (4, 5, 6, 7) the band (looped round A_1, A_2, A_3 as shown in Fig. 3) which crosses itself at i, j is not given, but the position of each crossing point is marked by a small cross. It should be

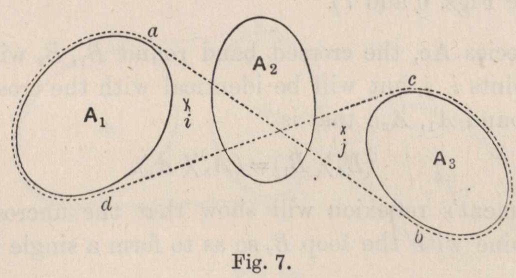

Fig. 7.

observed that in Fig. 5 (species Ab) j lies outside the crossed band round A_1, A_3; in Fig. 4 (species Aa) i and j lie in the same loop, and in Figs. 6, 7 (species Ac) i and j lie in opposite loops of the crossed band round A_1, A_3.

The discussion of species Aa is very simple; for it is clear that the conjunctive probability is

$$p_3 = (A_2 \times A_3) - (A_2 A_3)$$

since it is obviously impossible for a straight line to cut A_2 and A_3 without cutting A_1. Substituting this value for p_3 in formula (3) we obtain the disjunctive probability

$$\varpi_3 = (A_1) + (A_2) + (A_3) + (A_1 A_2) + (A_1 A_3) - (A_1 \times A_2) - (A_1 \times A_3).$$

The remaining two species belonging to class A may be discussed simultaneously; for we have in all the cases (see Fig. 8), using e, f to denote the

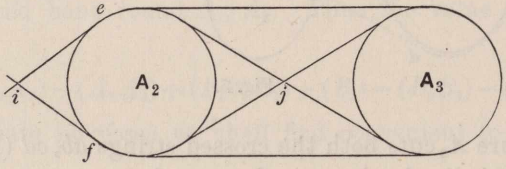

Fig. 8.

points of contact with the figure A_2 of the strings which cross at the point i (between A_1 and A_2),

$$(B_2 \times B_3) = (A_2 \times A_3) + fi + ie - ef,$$

$$(B_2 B_3) = (A_2 A_3) + fi + ie - ef,$$

so that $$(B_2 \times B_3) - (B_2 B_3) = (A_2 \times A_3) - (A_2 A_3).$$

Hence, for all the species of class A, formula (5) becomes

$$p_3 = (A_1 A_2 A_3) - (A_1 A_3) + (A_2 \times A_3) - (A_2 A_3) + (B_1 \times B_3) - (B_1 B_2) - (B_3).$$

In reducing the last three terms of this expression to a form which involves the lengths of bands round the A's, a slight difference arises between species Ab (in which, see Fig. 5, the point j and the figure A_1 are on the same side of the string ab) and species Ac (in which j and A_1 are on opposite sides of the string ab, see Figs. 6 and 7).

Thus, for species Ac, the crossed band round B_1, B_3 will not encounter either of the points i, j, but will be identical with the crossed band (*abcda*, Figs. 6 and 7) round A_1, A_3; that is

$$(B_3 \times B_1) = (A_3 \times A_1).$$

Moreover, a moment's reflexion will show that the uncrossed band round B_1, B_2 will combine with the loop B_3 so as to form a single band: in fact we have

$$(B_1 B_2) + (B_3) = D,$$

where D is the crossed band round A_1, A_3 with the loop which contains A_1 distended until it also contains A_2.

But in species Ab (see Fig. 9), let the points of contact with A_3 of the strings which cross at j (between A_2, A_3) be g, h; and let a string jk, in contact with A_1 at k, be stretched from j to the figure A_1: then

$$(B_1 \times B_3) = (A_1 \times A_3) + gj + jk + ka - ab - bg,$$

and

$$(B_1B_2) + (B_3) = D + gj + jk + ka - ab - bg,$$

where D is the band ($abgchjlmna$), derived from the crossed band ($abgcdna$) round A_1, A_3 by distending the loop which contains A_1 until it also contains A_2.

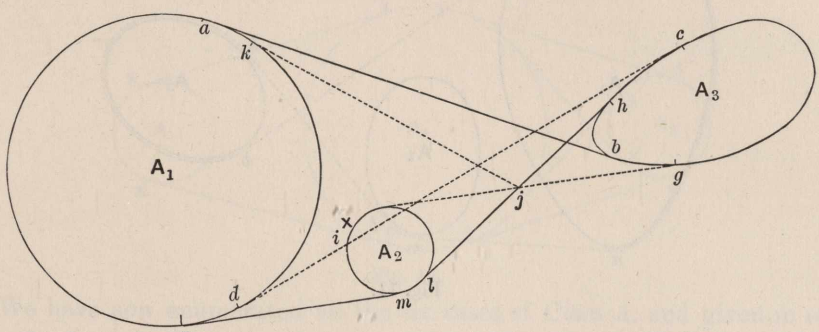

Fig. 9.

Hence $$(B_1 \times B_3) - (B_1B_2) - (B_3) = (A_1 \times A_3) - D,$$

and the general formula for the conjunctive probability (for class A) becomes

$$p_3 = (A_1A_2A_3) + (A_1 \times A_3) + (A_2 \times A_3) - (A_1A_3) - (A_2A_3) - D. \qquad (6)$$

Combining this with formula (3), which belongs to all cases of three figures, we obtain

$$\varpi_3 = (A_1) + (A_2) + (A_3) + (A_1A_2) + (A_1A_2A_3) - (A_1 \times A_2) - D.$$

The species Aa, Ab, Ac are distinguishable from one another by the difference in shape of the band D belonging to each. Thus in Aa the band

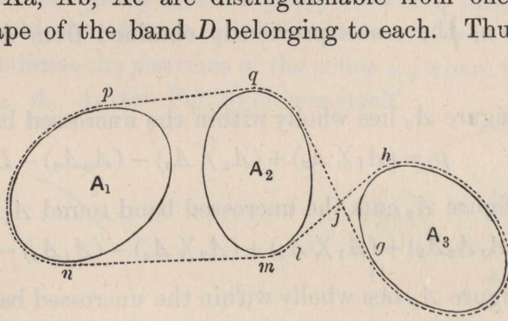

Fig. 10.

D is not distended at all, but is simply $(A_1 \times A_3)$; in Ab the loop containing A_1 is distended on one side only; and in Ac is distended on both sides (see Figs. 10 and 11). This difference in shape will be denoted by writing D_1

for D in the general formula when the species is Ab, and D_2 for D when the species is Ac.

The dotted bands (*pqjghjlmnp*) of Fig. 10, and (*abhlmna*) of Fig. 11 are what the dotted bands of Fig. 7 (species Ac) and Fig. 5 (species Ab) become, when the former is doubly and the latter singly distended.

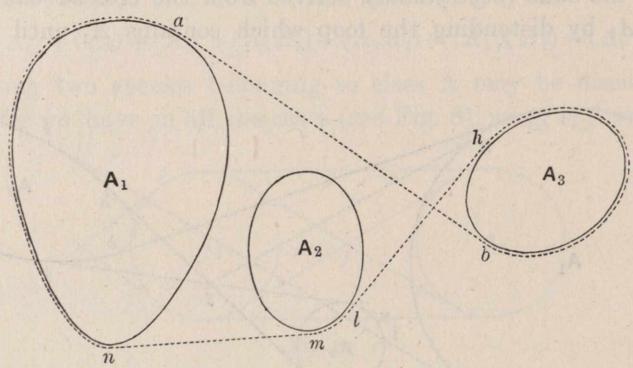

Fig. 11.

Varieties of the species in class A (namely one variety for Aa, two for Ab, and three for Ac, making 6 cases in all) occur when we consider the situation of the figure A_2 with respect to the uncrossed band round A_1, A_3. In all cases where A_2 lies wholly inside this band we have $(A_1A_2A_3) = (A_1A_3)$, so that in all such cases the general formula (6), which gives the conjunctive probability, becomes

$$p_3 = (A_1 \times A_3) + (A_2 \times A_3) - (A_2A_3) - D.$$

Aa. We have $D = (A_1 \times A_3)$

so that $p_3 = (A_2 \times A_3) - (A_2A_3)$

(the same as the result previously obtained from *à priori* considerations).

Ab. 1. The figure A_2 lies wholly within the uncrossed band round A_1, A_3

$$p_3 = (A_1 \times A_3) + (A_2 \times A_3) - (A_2A_3) - D_1.$$

Ab. 2. The figure A_2 cuts the uncrossed band round A_1, A_3

$$p_3 = (A_1A_2A_3) + (A_1 \times A_3) + (A_2 \times A_3) - (A_1A_3) - (A_2A_3) - D_1.$$

Ac. 1. The figure A_2 lies wholly within the uncrossed band round A_1, A_3.

Ac. 2. The figure A_2 cuts only one string of the uncrossed band round A_1, A_3. In these two cases the formulae which give p_3 are the same as in the corresponding varieties of Ab, except that D_2 takes the place of D_1.

Ac. 3. The figure A_2 cuts both strings of the uncrossed band round A_1, A_3. In this case the formula for the conjunctive probability

$$p_3 = (A_1 A_2 A_3) + (A_1 \times A_3) + (A_2 \times A_3) - (A_1 A_3) - (A_2 A_3) - D_2$$

becomes greatly simplified; for (see Fig. 12)

$$D_2 - (A_1 A_2 A_3) = rsjgu + vljht - rt - vu = (A_2 \times A_3) - (A_2 A_3)$$

so that $$p_3 = (A_1 \times A_3) - (A_1 A_3),$$

which is evidently true, since every straight line which cuts both A_1 and A_3 must also (in this case) cut A_2.

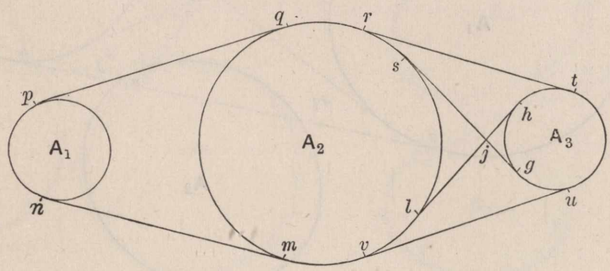

Fig. 12.

We have now enumerated all the six cases of Class A, and given in each case the formula for the conjunctive probability (from which, by means of formula (3), the disjunctive probability may be determined immediately). We proceed to the discussion of Class B.

In Class B (that is in the class where each figure lies entirely outside the crossed band round the other two) we recognize four species, and in one of them two varieties, making five cases in all. The enumeration is as follows.

Ba. There is one definite order of succession in which the three figures can be cut by a system of straight lines. There are two varieties of this species, namely

Ba. 1. The middle figure (A_2, see Fig. 13) lies wholly inside the uncrossed band round the other two. The small crosses in this figure, as in others, indicate the positions of the points i, j where the string looped round A_1, A_2, A_3 (see Fig. 3) crosses itself.

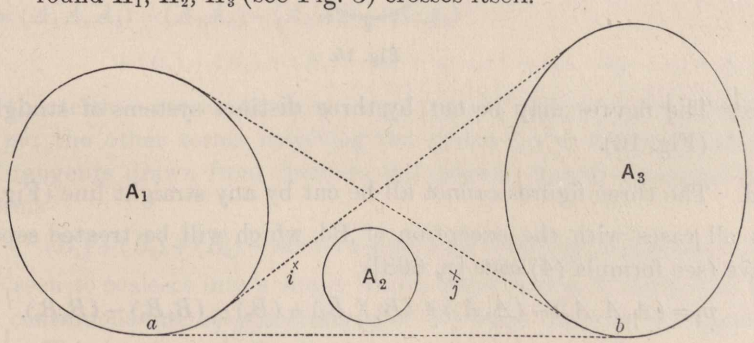

Fig. 13.

Ba. 2. The middle figure cuts the uncrossed band round the other two as shown in Fig. 14. In this, as in the preceding case, both i and j lie outside the crossed, but inside the uncrossed, band round A_1, A_3*.

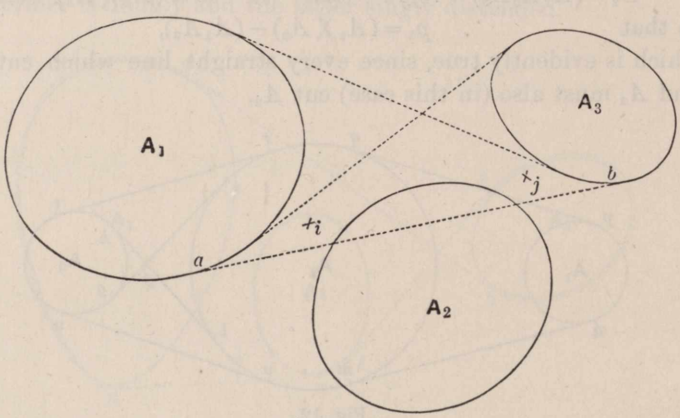

Fig. 14.

Bb. The figures may be cut in two different orders by two distinct systems of straight lines (see Fig. 15). One system of straight lines cuts the figures in the order A_1, A_2, A_3; the other system cuts them in the order A_3, A_1, A_2.

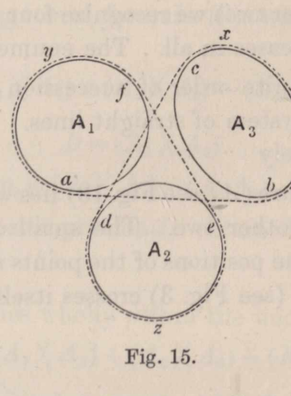

Fig. 15.

Bc. The figures may be cut by three distinct systems of straight lines (Fig. 16).

Bd. The three figures cannot all be cut by any straight line (Fig. 17).

In all cases with the exception of Bd, which will be treated separately, we have (see formula (4) *ante* [p. 668])

$$p_3 = (A_1 A_2 A_3) - (A_1 A_3) + (B_1 \times B_3) + (B_2) - (B_2 B_3) - (B_1 B_2).$$

* This circumstance enables us to discuss Ba. 1 and Ba. 2 simultaneously.

In Ba (see Fig. 18) we have

$$(B_2B_3) = (A_2A_3) + hi + ik - kc - cd - dh,$$
$$(B_1B_2) = (A_1A_2) + mj + jn - nf - fe - em,$$
$$(B_1 \times B_3) = (B_1) + (B_3) + ik - kc - cr - rj + jn - nf - fp - pi.$$

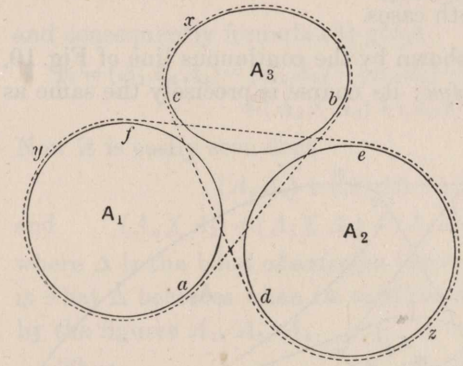

Fig. 16.

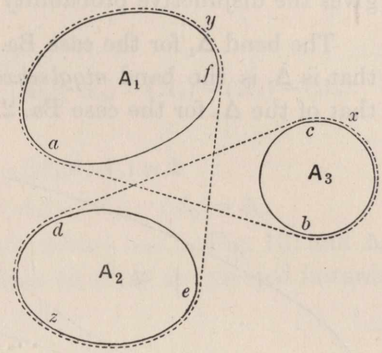

Fig. 17.

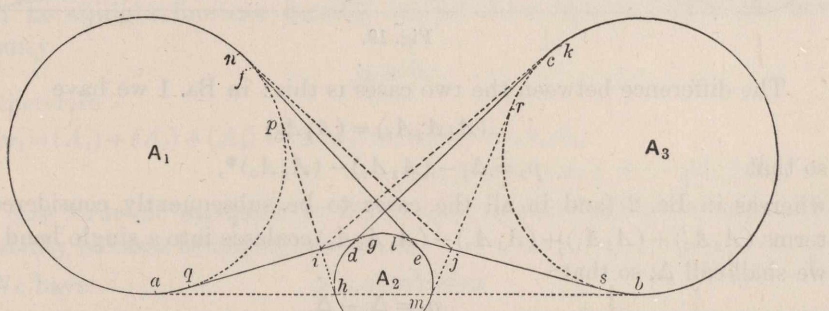

Fig. 18.

Substituting these values in the general expression for p_3, we obtain

$$p_3 = (A_1A_2A_3) - (A_2A_3) - (A_3A_1) - (A_1A_2)$$
$$+ (B_1) + (B_2) + (B_3) - mr - rc + cd + dh - hp - pf + fe + em$$

where the term $-mr$ comes from $-mj - rj$, and the term $-hp$ comes from $-hi - pi$; the other terms involving the points i, j or the points of contact k, n of tangents drawn from them to the original figures disappear in pairs. The terms

$$(B_1) + (B_2) + (B_3) - mr - rc + cd + dh - hp - pf + fe + em$$

will be seen to coalesce into a single band (whose course is marked in Fig. 18 by the continuous line *aqigljsbkcdhmefna*, all other lines in the figure being dotted). This band we shall call Δ_1.

Fig. 18 is drawn for the case Ba. 2, but the investigation of case Ba. 1 is precisely the same as that of Ba. 2. In both cases we find

$$p_3 = (A_1A_2A_3) - (A_2A_3) - (A_3A_1) - (A_1A_2) + \Delta_1$$

for the conjunctive probability, and consequently

$$\varpi_3 = (A_1) + (A_2) + (A_3) + (A_1A_2A_3) - (A_2 \times A_3) - (A_3 \times A_1) - (A_1 \times A_2) + \Delta_1$$

gives the disjunctive probability in both cases.

The band Δ_1 for the case Ba. 1 is shown by the continuous line of Fig. 19, that is Δ_1 is the band *atqglsvbxcdwuefya*: its course is precisely the same as that of the Δ_1 for the case Ba. 2.

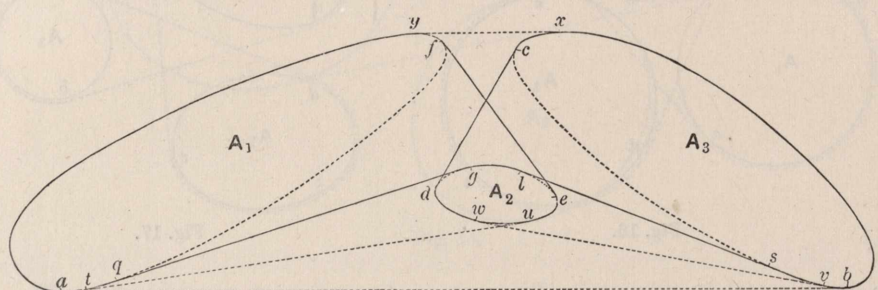

Fig. 19.

The difference between the two cases is this: in Ba. 1 we have

$$(A_1A_2A_3) = (A_1A_3)$$

so that $$p_3 = \Delta_1 - (A_1A_2) - (A_2A_3)^*,$$

whereas in Ba. 2 (and in all the cases to be subsequently considered) the terms $(A_2A_3) + (A_3A_1) + (A_1A_2) - (A_1A_2A_3)$ coalesce into a single band which we shall call Δ, so that

$$p_3 = \Delta_1 - \Delta.$$

The course of the band Δ is marked by the letters *abkcdhmefna* in Fig. 18. The band Δ_1 may be derived from Δ by supposing its rectilinear portion *ab* to be pressed inwards by the figure A_2 so as to occupy the position *aqglsb*.

The investigation of the case Bb proceeds on exactly the same lines as that of Ba. 2; we start from the same general formula and, by performing precisely similar work, obtain the result

$$p_3 = \Delta_2 - \Delta,$$

where (see Fig. 15) Δ is the band *abxcdzefya* whose course is indicated by dots, and Δ_2 is the band derived from Δ by supposing *two* of its rectilinear portions *ab, cd* to be pressed inwards by the figures A_1 and A_2.

* By an easy rearrangement of the bands the value of p_3 for this case may be expressed as the difference of the two bands, *atuelgdwvbxya* and *atqgleuwdglsvbxya* (see Fig. 19), derived from the uncrossed band *abxya* round A_1, A_3 by *twisting* its rectilinear portion *ab* right round A_2 in opposite directions.

In the case Bc (Fig. 16) the work is simplified by observing that each of the figures A_1, A_2, A_3 blocks the channel between the other two (that is, no straight line can pass between any two of them without cutting the third). Hence every straight line which cuts the uncrossed band round all the figures must cut one or more of them; that is

$$\varpi_3 = (A_1 A_2 A_3)$$

and consequently formula (3) gives

$$p_3 = (A_1 A_2 A_3) - (A_2 A_3) - (A_3 A_1) - (A_1 A_2) + (A_2 \times A_3) + (A_3 \times A_1) + (A_1 \times A_2) - (A_1) - (A_2) - (A_3).$$

Now it is easily seen that

$$(A_2 A_3) + (A_3 A_1) + (A_1 A_2) - (A_1 A_2 A_3) = \Delta$$

and $$(A_2 \times A_3) + (A_3 \times A_1) + (A_1 \times A_2) - (A_1) - (A_2) - (A_3) = \Delta_3$$

where Δ is the band *abxcdzefya* (shown by the dotted line in Fig. 16) and Δ_3 is what Δ becomes when its rectilinear portions *ab*, *cd*, *ef* are pressed inwards by the figures A_1, A_2, A_3.

Thus $$p_3 = \Delta_3 - \Delta.$$

The sole remaining case of three figures is Bd (Fig. 17), the case in which no straight line can possibly cut all three figures. In it we have obviously

$$p_3 = 0,$$

and therefore

$$\varpi_3 = (A_1) + (A_2) + (A_3) + (A_2 A_3) + (A_3 A_1) + (A_1 A_2) - (A_2 \times A_3) - (A_3 \times A_1) - (A_1 \times A_2).$$

This case forms no exception to the general rule for finding the conjunctive probability in cases belonging to class B.

We have $$\Delta = abxcdzefya$$

(that is, Δ is the dotted band of Fig. 17), and since this band is not pressed inwards by any of the figures the conjunctive probability according to the rule would be $\Delta - \Delta = 0$, which is right.

Having thus pointed out the general method of procedure, and illustrated it by treating in detail the case of three figures, it does not seem desirable to pursue the subject further in this direction for the present; but, before concluding, it may be worth while to notice that, in the general case of n limited right lines, the probabilities with which we have to do become Diophantine linear functions of the sides of the complete $2n$-gonal figure of which the n pairs of extremities of the lines are the angles. There will be a group of such linear functions depending on the mutual disposition of the n lines, but the number of formulae in any such group will be much greater than in the case of n general figures: for, when we pass from these to indefinitely narrow ovals, the portion of a definite band (appearing in any formula), partially surrounding any one of such ovals, may, according to the mutual disposition of their major axes, have in common with it an infinitesimal arc in some cases, in others an arc (to an infinitesimal *près*) equal to a circumference, and again in others to a semicircumference of the oval; which latter is ultimately the same as the length of the line whose double the complete circumference represents.

By way of illustration let us consider the question of two needles or limited straight lines rigidly connected. Neglecting the limiting cases, where one of the lines terminates in the other, there will remain three hypotheses:

A. The lines intersect.

B. The lines tend to intersect in a point external to each of them.

C. One of the lines tends towards a point lying within the other.

Let p_2 denote the chance of both the needles AB, CD being cut by one of the parallels, ϖ_2 the chance of one or other of them being cut: then we have the general formulae applicable to all cases

$$\varpi_2 = 2AB + 2CD - p_2,$$

p_2 = difference between the crossed and uncrossed bands round AB, CD.

A. When the lines intersect

$$\varpi_2 = AD + DB + BC + CA,$$
$$p_2 = 2AB + 2CD - AD - DB - BC - CA.$$

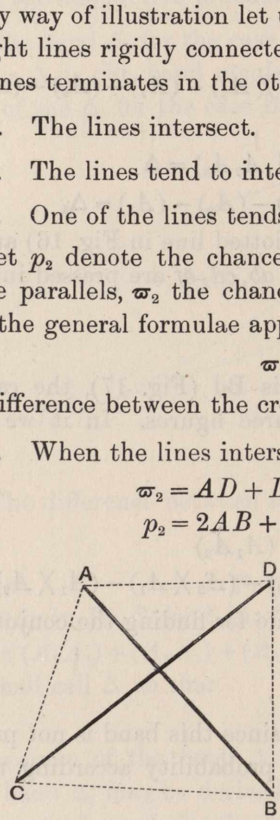

Fig. 20.

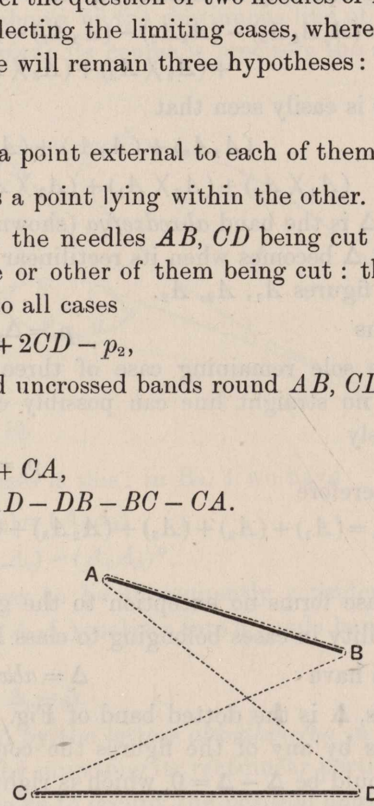

Fig. 21.

B. When the lines tend to intersect in a point external to each of them

$$p_2 = (AB + BC + CD + DA) - (AB + BD + DC + CA)$$
$$= BC - CA + AD - DB^*,$$
$$\varpi_2 = 2AB + 2CD - BC + CA - AD + DB.$$

* Imagine a string passing from B to C, from C to A, from A to D, and from D to B. This string cannot be kept tight unless fastened by pins at A, B, C, D. Inserting the necessary pins and tightening the string, we agree to consider the consecutive portions of the string as alternately positive and negative.

On these suppositions p_2 is the algebraical length of the band $BCADB$ stretched round the pins. The method of representation by means of pinned bands may be extended to the case of two (or any number of) general figures.

C. When one of the lines tends towards a point lying within the other

$$p_2 = (2AB + BC + CD + DB) - (AC + CD + DA)$$
$$= 2AB + BC - CA - AD + DB,$$
$$\varpi_2 = 2CD - BC + CA + AD - DB.$$

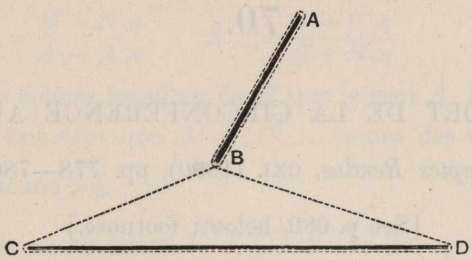

Fig. 22.

The complexity of cases for three right lines is such as would require a separate study even to obtain a perfect enumeration of them; consequently I shall leave it to others to pursue the subject further whether as regards principles or details. I will only add that the ascertainment of the general law that the formulae contain no other arguments than lengths of tight endless bands variously drawn round the given contours appears to me a distinct step achieved in the prosecution of this extensive theory, and one that is far from being obvious *à priori*. Buffon's problem of the needle, it will be seen, has now expanded into a problem of n needles rigidly connected, which may be treated as a corollary to that of *n entirely separate* general contours, the mode of solution of which, it is believed, has been sufficiently indicated in the investigations which form the subject of this memoir.

POSTSCRIPTUM. Since the above was set up in print my attention has been called to the fact that the extension of Barbier's theorem referred to on p. [664] is due to Prof. Crofton and is given by him in his celebrated paper on the *Theory of Local Probability* contained in the *Philosophical Transactions* for 1868. Strange to say, no reference to this, so far as I can find, is made in Czuber's treatise. It is the more singular that I should have overlooked the fact inasmuch as it was an outcome of conversations with myself, when Prof. Crofton was serving under me in the Royal Military Academy at Woolwich, that he was put upon the track of investigations in local probability in which he has since earned for himself so great and well merited celebrity. It may be added that Prof. Crofton seems to have written in entire ignorance of Barbier's discovery as he makes no allusion to it in his paper.

It is indeed a romantic incident in mathematical history that Buffon's problem of the needle should have led up (as is undoubtedly the case) to Crofton's new and striking theorems in the integral calculus reproduced in Bertrand's *Calcul intégral*.