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ON THE SKEW SURFACE OF THE THIRD ORDER.

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THE skew surface of the third order, or "cubic scroll" (disregarding a certain special form), may be considered as generated by a line which always passes through three directrices; viz., a plane cubic having a node, and two lines, one of them meeting the cubic in the node, the other of them meeting the cubic in an ordinary point. The analytical investigation possesses some interest as an illustration of the analytical theory of skew surfaces in general.

Take for the equation of the cubic

$$(\alpha^3+\beta^3)\,xy-(x^3+y^3)\,\alpha\beta=0,$$

which belongs to a cubic having a node at the origin, and passing through the point (α, β); and for the equations of the two lines

$$(x-mz=0,\quad y-nz=0),$$
$$(x-\ \alpha\ =0,\quad y-\ \beta\ =0).$$

Then, (X, Y, Z) being current coordinates, the equations of the generating line will be

$$X=x+AZ,$$
$$Y=y+BZ\,;$$

when this meets the line $(X-mZ=0,\ Y-nZ=0)$, we have

$$mZ=x+AZ,$$
$$nZ=y+BZ,$$

and thence

$$x(n-B)-y(m-A)=0\,;$$

or, what is the same thing,

$$nx - my - Bx + Ay = 0:$$

and when it meets the line $(X - \alpha = 0,\ Y - \beta = 0)$, we have

$$\alpha = x + AZ,$$
$$\beta = y + BZ;$$

and thence

$$B(x - \alpha) - A(y - \beta) = 0.$$

We have thus the system of equations

$$(\alpha^3 + \beta^3)\, xy - (x^3 + y^3)\, \alpha\beta = 0,$$

$$X = x + AZ,$$
$$Y = y + BZ,$$
$$nx - my - Bx + Ay = 0,$$
$$B(x - \alpha) - A(y - \beta) = 0;$$

from which, eliminating (A, B, x, y), we obtain the equation of the surface.

Writing in the last equation

$$B = s(x - \alpha), \quad A = s(y - \beta)$$

(values which give $Bx - Ay = -s(\beta x - \alpha y)$), we find

$$X + \alpha sZ = (1 + sZ)\, x,$$
$$Y + \beta sZ = (1 + sZ)\, y,$$
$$(n + \beta s)\, x - (m + \alpha s)\, y = 0;$$

whence also

$$(n + \beta s)(X + \alpha sZ) - (m + \alpha s)(Y + \beta sZ) = 0,$$

that is

$$nX - mY + (n\alpha - m\beta)\, sZ + s(\beta X - \alpha Y) = 0;$$

or eliminating s from this equation and the two equations

$$x - X + Z(x - \alpha)\, s = 0,$$
$$y - Y + Z(y - \beta)\, s = 0,$$

we have

$$\{(n\alpha - m\beta)\, Z + \beta X - \alpha Y\}(x - X) - Z(x - \alpha)(nX - mY) = 0,$$
$$\{(n\alpha - m\beta)\, Z + \beta X - \alpha Y\}(y - Y) - Z(y - \beta)(nX - mY) = 0;$$

these give

$$\Omega x = X\{(n\alpha - m\beta)\, Z + \beta X - \alpha Y\} - \alpha Z(nX - mY)$$
$$= -mZ\beta X + X(\beta X - \alpha Y) + mZ\alpha Y$$
$$= (X - mZ)(\beta X - \alpha Y),$$

and

$$\begin{aligned}\Omega y &= Y\{(n\alpha - m\beta)Z + \beta X - \alpha Y\} - \beta Z(nX - mY)\\ &= nZ\alpha Y + Y(\beta X - \alpha Y) - nZ\beta X\\ &= (Y - nZ)(\beta X - \alpha Y),\end{aligned}$$

where

$$\begin{aligned}\Omega &= (n\alpha - m\beta)Z + (\beta X - \alpha Y) - Z(nX - mY)\\ &= \beta(X - mZ) - \alpha(Y - nZ) - Z\{n(X - mZ) - m(Y - nZ)\}\\ &= (\beta - nZ)(X - mZ) - (\alpha - mZ)(Y - nZ);\end{aligned}$$

so that

$$x = \frac{(X - mZ)(\beta X - \alpha Y)}{(\beta - nZ)(X - mZ) - (\alpha - mZ)(Y - nZ)},$$

$$y = \frac{(Y - nZ)(\beta X - \alpha Y)}{(\beta - nZ)(X - mZ) - (\alpha - mZ)(Y - nZ)},$$

which equations give the coordinates (x, y) of the point in which the generating line through the point (X, Y, Z) of the surface meets the cubic

$$(\alpha^3 + \beta^3)xy - (x^3 + y^3)\alpha\beta = 0.$$

Substituting these values of (x, y) in the equation of the cubic, we obtain the equation

$$\begin{aligned}(\alpha^3 + \beta^3)(X - mZ)(Y - nZ)\{(\beta - nZ)(x - mZ) - (\alpha - mZ)(Y - nZ)\}&\\ - \alpha\beta(\beta X - \alpha Y)\{(X - mZ)^3 + (Y - nZ)^3\} &= 0;\end{aligned}$$

or, as it may be written,

$$\begin{aligned}(\alpha^3 + \beta^3)(X - mZ)(Y - nZ)\{\beta(X - mZ) - \alpha(Y - nZ)\}&\\ + (\alpha^3 + \beta^3)(X - mZ)(Y - nZ)Z(mY - nZ)&\\ - \alpha\beta(\beta X - \alpha Y)\{(X - mZ)^3 + (Y - nZ)^3\} &= 0.\end{aligned}$$

This equation contains, however, the extraneous factor

$$\beta(X - mZ) - \alpha(Y - nZ),$$

which, equated to zero, gives the equation of the plane through the node and the line $(x - mz = 0,\ y - nz = 0)$. In fact, assuming

$$\begin{aligned}(\alpha^3 + \beta^3)(X - mZ)(Y - nZ)Z(mY - nZ) - \alpha\beta(\beta X - \alpha Y)\{(X - mZ)^3 + (Y - nZ)^3\}&\\ = \{\beta(X - mZ) - \alpha(Y - nZ)\}\,\Phi(X, Y, Z),&\end{aligned}$$

it will presently be shown that Φ is an integral function. Hence, omitting the factor in question, we have

$$(\alpha^3 + \beta^3)(X - mZ)(Y - nZ) + \Phi(X, Y, Z) = 0,$$

which is the equation of the surface. It only remains to find Φ: writing for this purpose $X+mZ$, $Y+nZ$ in the place of X, Y, respectively, and putting for a moment

$$\Phi(X+mZ,\ Y+nZ,\ Z)=\Phi',$$

we have

$$(\alpha^3+\beta^3)\,XYZ\,(mY-nZ)-\alpha\beta\,\{\beta(X+mZ)-\alpha(Y+nZ)\}\,(X^3+Y^3)=(\beta X-\alpha Y)\,\Phi';$$

that is

$$(\beta X-\alpha Y)\,\Phi'=Z\,\{(\alpha^3+\beta^3)\,XY\,(mY-nZ)-(X^3+Y^3)\,\alpha\beta\,(m\beta-n\alpha)\}-(\beta X-\alpha Y)\,\alpha\beta\,(X^3+Y^3);$$

or, effecting the division,

$$\Phi'=Z\,\{(X^2\alpha-Y^2\beta)(\alpha n-\beta m)-XY(\alpha^2m+\beta^2n)\}-\alpha\beta\,(X^3+Y^3),$$

and then writing $X-mZ$, $Y-nZ$ in the place of X, Y respectively, we have

$$\Phi(X,\ Y,\ Z)=Z\,\{((X-mZ)^2\alpha-(Y-nZ)^2\beta)\,(\alpha n-\beta m)$$
$$-(X-mZ)(Y-nZ)(\alpha^2m+\beta^2n)\}-\alpha\beta\,\{(X-mZ)^3+(Y-nZ)^3\}.$$

Hence, finally, the equation of the surface is

$$(\alpha^3+\beta^3)(X-mZ)(Y-nZ)-\alpha\beta\,\{(X-mZ)^3+(Y-nZ)^3\}$$
$$+Z\,\{((X-mZ)^2\alpha-(Y-nZ)^2\beta)\,(\alpha n-\beta m)-(X-mZ)(Y-nZ)(\alpha^2m+\beta^2n)\}=0,$$

which is, as it should be, of the third order.

Arranging in powers of Z and reducing, the equation is found to be

$$(\alpha^3+\beta^3)\,XY-\alpha\beta\,(X^3+Y^3)$$
$$+Z\,\{-(\alpha^3+\beta^3)(mY+nX)+(X^2\alpha+Y^2\beta)(m\beta+n\alpha)+\alpha\beta\,(mX^2+nY^2)-(\alpha^2m+\beta^2n)\,XY\}$$
$$+Z^2\,\{mn\,(\alpha^3+\beta^3-\alpha^2X-\beta^2Y)(\beta n^2-\alpha m^2)(\beta X-\alpha Y)\}=0.$$

The first form puts in evidence the nodal line

$$(X-mZ=0,\quad Y-nZ=0),$$

and the second form puts in evidence the simple line

$$(X-\alpha=0,\quad Y-\beta=0).$$

But to obtain a more convenient form, write for a moment $X-mZ=P$, $Y-nZ=Q$; the equation is

$$(\alpha^3+\beta^3)\,PQ-\alpha\beta\,(P^3+Q^3)+Z\,\{(P^2\alpha-Q^2\beta)(n\alpha-m\beta)-PQ\,(m\alpha^2+n\beta^2)\}=0,$$

or, as this may be written,

$$(\alpha^3+\beta^3)\,PQ+(\alpha^2P-\beta^2Q)\,Z\,(Pn-Qm)+\alpha\beta\,\{-P^3-Q^3-Z\,(mP^2+nQ^2)\}=0;$$

or, observing that $X = P + mZ$, $Y = Q + nZ$, and thence

$$PY - QX = Z(Pn - Qm), \quad XP^2 + QY^2 = P^3 + Q^3 + Z(mP^2 + nQ^2),$$

the equation becomes

$$(\alpha^3 + \beta^3)\,PQ + (\alpha^2 P - \beta^2 Q)(PY - QX) - \alpha\beta\,(P^2 X + Q^2 Y) = 0,$$

or, what is the same thing,

$$(\alpha P^2 - \beta Q^2)(\alpha Y - \beta X) + PQ\,(\alpha^3 + \beta^3 - \alpha^2 X - \beta^2 Y) = 0\,;$$

whence, making a slight change in the form, and restoring for P, Q their values, the equation is

$$\{\alpha(X - mZ)^2 - \beta(Y - nZ)^2\}\{\alpha(Y - \beta) - \beta(X - \alpha)\}$$
$$- (X - mZ)(Y - nZ)\{\alpha^2(X - \alpha) + \beta^2(Y - \beta)\} = 0,$$

a form which puts in evidence as well the simple line $(X - \alpha = 0,\ Y - \beta = 0)$ as the nodal line $(X - mZ = 0,\ Y - nZ = 0)$.

If $Z = 0$, we have

$$(\alpha X^2 - \beta Y^2)(\alpha Y - \beta X) - XY\{\alpha^2(X - \alpha) + \beta^2(Y - \beta)\} = 0,$$

which is in fact the cubic curve $(\alpha^3 + \beta^3)XY - \alpha\beta(X^3 + Y^3) = 0$.

Reverting to a former system of equations

$$nx - my - Bx + Ay = 0,$$
$$B(x - \alpha) - A(y - \beta) = 0,$$

or, as these may be written,

$$Bx - Ay = nx - my,$$
$$B\alpha - A\beta = nx - my,$$

we find

$$B(\beta x - \alpha y) = (\beta - y)(nx - my),$$
$$A(\beta x - \alpha y) = (\alpha - x)(nx - my)\,;$$

so that we have

$$X = x + \frac{(\alpha - x)(nx - my)}{\beta x - \alpha y}\,Z,$$

$$Y = y + \frac{(\beta - y)(nx - my)}{\beta x - \alpha y}\,Z,$$

as the equations of the generating line which passes through the point (x, y) of the cubic curve.

2, *Stone Buildings, W.C., October* 28, 1862.