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THEOREMS RELATING TO THE CANONIC ROOTS OF A BINARY QUANTIC OF AN ODD ORDER.

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I CALL to mind Professor Sylvester's theory of the canonical form of a binary quantic of an odd order; viz., the quantic of the order $2n+1$ may be expressed as a sum of a number $n+1$ of $(2n+1)$th powers, the roots of which, or say the *canonic roots* of the quantic, are to constant multipliers *près* the factors of a certain covariant derivative of the order $(n+1)$, called the *Canonizant*. If, to fix the ideas, we take a quintic function, then we may write

$$(a, b, c, d, e, f \between x, y)^5 = A\,(lx+my)^5 + A'\,(l'x+m'y)^5 + A''\,(l''x+m''y)^5$$

(it would be allowable to put the coefficients A each equal to unity; but there is a convenience in retaining them, and in considering that a canonic root $lx+my$ is only given as regards the ratio $l:m$, the coefficients l, m remaining indeterminate); and then the canonic roots $(lx+my)$, &c. are the factors of the Canonizant

$$\begin{vmatrix} y^3, & -y^2x, & yx^2, & -x^3 \\ a\,, & b\,, & c\,, & d \\ b\,, & c\,, & d\,, & e \\ c\,, & d\,, & e\,, & f \end{vmatrix}.$$

It is to be observed that this reduction of the quantic to its canonical form, i.e. to a sum of a number $n+1$ of $(2n+1)$th powers, is a *unique* one, and that the quantic cannot be in any other manner a sum of a number $n+1$ of $(2n+1)$th powers.

Professor Sylvester communicated to me, under a slightly less general form, and has permitted me to publish the following theorems:

1. If the second emanant $(X\partial_x + Y\partial_y)^2 U$ has in common with the quantic U a single canonic root, then all the canonic roots of the emanant are canonic roots of the quantic; and, moreover, if the remaining canonic root of the quantic be $rx+sy$, then (X, Y), the facients of emanation, are $=(s, -r)$, or, what is the same thing, they are given by the equation

$$\text{canont. } U \ (X,\ Y \text{ in place of } x,\ y) = 0.$$

In fact, considering, as before, the quintic $U = (a, b, c, d, e, f \between x, y)^5$, we have

$$U = A\,(lx+my)^5 + A'\,(l'x+m'y)^5 + A''\,(l''x+m''y)^5,$$

and thence

$$(X\partial_x + Y\partial_y)^2 U = B\,(lx+my)^3 + B'\,(l'x+m'y)^3 + B''\,(l''x+m''y)^3,$$

if for shortness

$$B = 6\,.\,5\,(lX+mY)^2 A, \ \&\text{c}.$$

Suppose $(X\partial_x + Y\partial_y)^2 U$ has in common with U the canonic root $lx+my$, then

$$(X\partial_x + Y\partial_y)^2 U = C\,(lx+my)^3 + C'\,(px+qy)^3,$$

and thence

$$B'\,(l'x+m'y)^3 + B''\,(l''x+m''y)^3 = (C-B)\,(lx+my)^3 + C'\,(px+qy)^3,$$

which must be an identity; for otherwise we should have the same cubic function expressed in two different canonical forms. And we may write

$$B' = C', \quad l'x+m'y = px+qy, \quad B'' = 0, \quad C = B,$$

and then we have

$$(X\partial_x + Y\partial_y)^2 U = B\,(lx+my)^3 + B'\,(l'x+m'y)^3;$$

so that all the canonic roots of the emanant are canonic roots of the quantic. Moreover, the condition $B''=0$ gives $l''X + m''Y = 0$, that is, $X : Y = m'' : -l''$, or writing $rx+sy$ instead of $l''x+m''y$, $X : Y = s : -r$; and the system is

$$U = A\,(lx+my)^5 + A'\,(l'x+m'y)^5 + A\,(rx+sy)^5,$$

$$(s\partial_x - r\partial_y)^2\, U = B\,(lx+my)^3 + B'\,(l'x+m'y)^3,$$

which proves the theorem.

2. The two functions, canont. U, canont. $(X\partial_x + Y\partial_y)^2 U$, have for their resultant $\{\text{canont. } U\ (X,\ Y \text{ in place of } x,\ y)\}^{2n}$, if $2n+1$ be the order of U.

In fact, in order that the equations

$$\text{canont. } U = 0, \quad \text{canont. } (X\partial_x + Y\partial_y)^2 U = 0,$$

may coexist, their resultant must vanish; and conversely, when the resultant vanishes, the equations will have a common root. Now if the equation canont. $(X\partial_x + Y\partial_y)^2 U = 0$ has a common root with the equation canont. $U = 0$, all its roots are roots of canont. $U = 0$; and, moreover, if $rx + sy = 0$ be the remaining root of canont. $U = 0$, then $X : Y = s : -r$, that is, we have

$$\text{canont. } U \ (X, Y \text{ in place of } x, y) = 0;$$

or the resultant in question can only vanish if the last-mentioned equation is satisfied. It follows that the resultant must be a power of the *nilfactum* of the equation; and observing that canont. U is of the form $(a, \ldots)^{n+1}(x, y)^{n+1}$, i.e. that it is of the degree $n+1$ as well in regard to the coefficients as in regard to the variables (x, y), it is easy to see that the resultant is of the degree $2n(n+1)$ as well in regard to the coefficients as in regard to (X, Y); that is, we have $2n$ as the index of the power in question.

3. In particular, if $Y = 0$, the theorem is that the resultant of the functions canont. U, canont. $\partial_x{}^2 U$ is equal to the $2n$th power of the first coefficient of canont. U.

Thus for $n = 1$, that is, for the cubic function $(a, b, c, d \between x, y)^3$, we have

$$\text{canont. } U = \begin{vmatrix} y^2, & -xy, & x^2 \\ a, & b, & c \\ b, & c, & d \end{vmatrix} = (ac - b^2, \ ad - bc, \ bd - c^2 \between x, y)^2,$$

$$\text{canont. } \partial_x{}^2 U = \begin{vmatrix} y, & -x \\ a, & b \end{vmatrix} = ax + by;$$

and the resultant of the two functions is

$$= (ac - b^2, \ ad - bc, \ bd - c^2 \between b, -a)^2$$
$$= -(ac - b^2)^2,$$

which verifies the theorem.

The theorems were, in fact, given to me in relation to the quantic U and the second differential coefficient $\partial_x{}^2 U$; but the introduction instead thereof of the second emanant $(X\partial_x + Y\partial_y)^2 U$ presented no difficulty.

2, *Stone Buildings, W.C., February* 16, 1863.