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ON CONTOUR AND SLOPE LINES.

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It is, I think, interesting as a question of topography, to consider the general configuration of a system of contour lines and steepest or slope lines (*lignes de niveau* and *lignes de la plus grande pente*). Imagine, to fix the ideas, a mountainous island, the exterior or sea-level contour line being consequently a closed curve; the case where any contour line is a curve cutting itself is an important one, which will be considered; but disregarding it for the moment, and excluding (as I do throughout) a curve which cuts itself from the notion of a closed curve, the entire contour line corresponding to a given elevation will be either a single closed curve, or it will consist of two or more separate closed curves, in the latter case each of these may be considered as being by itself a contour line, and we may therefore say that the contour line is in general a closed curve. It may happen that the elevation of a given contour line is a maximum or minimum; in other words, that the consecutive curve without the given contour line and that within it are each of them higher or each of them lower than the given contour line; but this is a speciality which need not be particularly attended to; in general the consecutive curve without the given contour line will be lower, and that within it higher than the given contour line, in which case the tract bounded by the contour line is an elevation (hill, table-land, or mountain, as the case may be); or else the consecutive curve without the given contour line is higher, and that within it lower than the given contour line; in which case the tract bounded by the contour line is a depression. But there may be within the contour line bounding an elevation, spaces lower than the bounding line, and within the contour line bounding a depression, spaces higher than the bounding line. A depression usually contains water, and indeed is filled so as to overflow, in which case there is a lake with an outlet; if the depression is not filled to overflowing, the lake will have no outlet. The contour line bounding an elevation may become indefinitely small and ultimately reduce itself to a point, which is a *summit;* the contour line bounding a depression may in like manner become indefinitely small, and ultimately reduce itself to a point, which is what I call an *immit.* A summit is a point of maximum elevation (though of course there may be summits, or even immits, which are higher); an immit is a point of minimum elevation. But there are besides, as at the heads of passes, points where the surface is horizontal, but where the elevation is neither a maximum nor a minimum; you descend backwards and forwards, but ascend right and left: I will for the present purpose call this kind of point a *knot.* And this leads to the consideration of a contour line which cuts itself: the point where this happens is in fact a knot, or geometrically the knot is a node or double point on the contour line. It may be assumed that the contour line through a knot does not pass through any other knot; for although there may be neighbouring passes of precisely the same elevation, yet the general configuration of the country will not be altered by giving a slight difference of elevation to such passes: the effect of this alteration is to distribute among contour lines of slightly different elevations (one to each line) the different knots which would otherwise occur upon one and the same contour line. The contour line through a knot cuts itself therefore at this point only: such contour line is either a figure of eight, or as I will term it, an *outloop* curve; or else it is the figure formed by the union (so as to give rise to a node or double point) of two closed curves, one of which lies within the other of them; this I call an *inloop* curve. An outloop curve consists of two loops; the spaces within these may also be spoken of as the loops. An inloop curve consists of an outer and an inner loop; the space within the inner loop may be spoken of as the inner loop, that between the two loops as the *lune.* It usually happens, and to fix the ideas I will assume, that for an outloop curve each of the loops is an elevation: this is the case of two mountain summits connected by a ridge or col, the lowest point whereof, or head of the pass, is the knot on the outloop contour line through this point. And in like manner, that for an inloop curve the lune is an elevation, the inner loop a depression; and that the outer loop, considered as a portion of the contour line, is higher than the consecutive exterior contour line. This is the case of a lake having an outlet; if the lake were dry, the passage up stream into the bed of it would be over a ridge, col, or barrier, the lowest point whereof, or point of outlet for the water of the lake, is the knot on the inloop contour line passing through this point, the shore of the lake being of course the inner loop of this contour line, and the waters being retained by means of the raised ground within the lune between the two loops of the contour line.

The slope lines cut at right angles the contour lines; and this property applies also to the projections of the two systems of lines; so that the two systems of lines delineated *in plano* intersect at right angles. Consider the contour lines which are closed curves surrounding a given summit or immit; the exterior contour line is intersected at each of its points by a slope line; and all these slope lines must, it is clear, intersect all the interior contour lines, and ultimately unite at the interior summit or immit. In order to see more distinctly the form of the system of slope lines, it is to be noticed that, if (as is in general the case) the indicatrix at the summit or immit be an ellipse, the contour lines in the immediate neighbourhood thereof will be a system of similar and similarly situated concentric ellipses, the major and minor axes whereof correspond respectively with the directions of least and greatest curvature; the equation of any orthogonal trajectory of the ellipses, if a, b are the semi-axes, major and minor, of any one of them, is $y^{b^2} = Cx^{a^2}$; and unless $C = \infty$, the curve represented by this equation touches the axis of x, which is the direction of least curvature; if however $C = \infty$, then the equation becomes $x = 0$, and the curve touches the axis of y, which is the direction of greatest curvature. Hence in general at a summit or immit the slope curves all, except one (which is a limiting case) touch the line which is the direction of least curvature. The only exception is when the summit or immit is an umbilicus—the indicatrix is then a circle; the contour lines in the immediate neighbourhood of this point are concentric circles, and the slope lines pass in all directions through the summit or immit.

The indicatrix at a knot is in general a hyperbola, and consequently the contour lines in the neighbourhood of a knot are similar and similarly situated concentric hyperbolas; and if a, b are the semi-axes of one of these hyperbolas, the equation of an orthogonal trajectory is $x^{a^2}y^{b^2} = C$: and when this passes through the knot, $C = 0$; and therefore either $x = 0$ or else $y = 0$; there are consequently through the knot only two slope lines, which bisect the angles made by the two branches of the contour line and intersect each other at right angles. The slope lines through a knot may be termed ridge and course lines: and for one of these—the ridge line—the knot is a point of minimum elevation; for the other of them—the course line—the knot is a point of maximum elevation. But this requires some further development. To fix the ideas, consider the case where the contour line is an outloop curve, the loops being each of them elevations. The slope line through the knot, and which lies within the two loops, would be, according to the definition, a ridge line. Suppose that the contour lines within one of the loops are closed curves surrounding a summit, the ridge line will, it is clear, cut all these curves and ultimately arrive at the summit. But if the contour lines within the loop are not all of them closed curves; if, for instance, they are first closed curves, then an outloop curve, and within each of the loops of this, closed curves surrounding a summit, then it may happen that the above-mentioned ridge line will pass through the knot of the inner outloop curve: and with respect to this knot, it will be, not a ridge line, but a course line; so that the slope line in question cannot be spoken of *simpliciter* either as a ridge line or as a course line, but it is the one or the other *quoad* the knot in reference to which it is considered; and, considered by itself, it can only be spoken of as a ridge-or-course line. The case just referred to is, however, an exceptional one; in general the slope line in question would not pass through the knot of the inner outloop curve, but would cut one of the loops of this curve, and then cutting all the contour lines within such loop, arrive at last at the summit within such loop. And when the ridge line has once arrived at a summit, there is little meaning in continuing it further, and it may be considered as ending there; in fact there are through the summit an infinity of slope lines, all of them (except in the case where the summit is an umbilicus) coincident in direction with the ridge line, and consequently the ridge line may, without graphical discontinuity, be considered as proceeding along any one of these lines indifferently;

and although, when the surface is a geometrical one capable of being represented by an equation, there would be geometrically one of these slope lines which could be identified as the continuation of the ridge line, there would be no advantage in making this identification. Hence it may be considered that in general a ridge line passes from summit to summit, through a single intervening knot which is a point of minimum elevation on the ridge line; and in like manner, that in general a course line passes from immit to immit through a single intervening knot which is a point of maximum elevation on the course line; it need not be considered as an exception when, as is frequently the case, the course line arrives at the sea-level contour line without previously reaching an immit. It is to be noticed that a ridge line or a course line may commence and terminate at one and the same summit or immit, and thus form a closed curve.

The ridge lines, as above defined, determine the watershed. In the case of an isolated conical or dome-shaped mountain, and in general when the contour lines are all of them closed curves, there is no definable watershed; but in the case of a chain of mountain summits, the watershed runs from summit to summit through the heads of the passes over the connecting cols, i.e. it is made up of a series of ridge lines each extending from a summit to a summit through an intervening knot. And the course lines are, as nearly as may be, the beds of the streams which flow from the heads of the passes down the lateral valleys. The ridge line and the course line respectively are, I believe, the so-called *ligne de faîte* and *ligne de thalweg*.

2, *Stone Buildings, W.C., July* 20, 1859.