257.

ON THE CUBIC CENTRES OF A LINE WITH RESPECT TO THREE LINES AND A LINE.

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CONSIDER a line L in relation to the three lines X, Y, Z and the line I: through the point of intersection of the lines X, L, draw any line meeting the lines I, Y, Z, and let the harmonic of the intersection with I, in relation to the intersections with Y, Z, be ξ; then the locus of the point ξ is a conic passing through the points YI, ZI, YZ.

If, in like manner, through the point of intersection of the lines Y, L, there is drawn any line meeting the lines I, Z, X, and the harmonic of the intersection with I, in relation to the intersections with Z, X, is called η, the locus of the point η is a conic passing through the points ZI, XI, ZX.

And so, if through the point of intersection of the lines Z, L there is drawn any line meeting the lines I, X, Y, and the harmonic of the intersection with I, in relation to the intersections with X, Y, is called ζ, then the locus of ζ is a conic passing through the points XI, YI, XY.

The pairs of conics, viz. the second and third, third and first, first and second conics, have obviously in common the points XI, YI, ZI respectively. They besides intersect all three of them in three points, which may be termed the *cubic centres* of the line L in relation to the lines X, Y, Z and the line I.

The line L may be such that two of the three cubic centres coincide; the locus of the coincident centres is in this case a conic which touches the lines X, Y, Z harmonically in regard to the line I; that is, it touches each of the three lines in the point which is the harmonic of its intersection with I in relation to its intersections with the other two lines.

Except that the line I is there taken to be infinity, the foregoing theorems occur in Plücker's *System der analytischen Geometrie* (Berlin, 1835), p. 177 *et seq.*; and they play an important part in his classification of curves of the third order (see p. 220 *et seq.*). It is, I think, an omission that he has not sought for the curve which is the envelope of the line L in the above-mentioned case of the two coincident centres: I find that the envelope is a curve of the fourth order, having four-pointic contact with the lines X, Y, Z harmonically in regard to the line I; viz., if the equations of the lines X, Y, Z are $x=0$, $y=0$, $z=0$ respectively, and the equation of the line I is $x+y+z=0$, then the equation of the envelope in question is

$$\sqrt[4]{x}+\sqrt[4]{y}+\sqrt[4]{z}=0,$$

a result which is also interesting as exhibiting a geometrical construction of the curve represented by this equation.

The investigation of the series of theorems is as follows; taking

$$\begin{array}{rll} x=0 & \text{for the equation of} & X, \\ y=0 & \quad\quad ,, & Y, \\ z=0 & \quad\quad ,, & Z, \\ x+\ y+\ z=0 & \quad\quad ,, & I, \\ \lambda x+\mu y+\nu z=0 & \quad\quad ,, & L, \end{array}$$

then, first, in order to find the curve which is the locus of ξ, the coordinates of the point XL are given by $x:y:z=0:\nu:-\mu$; or if, as it is convenient to do, we take X, Y, Z (instead of x, y, z) for current coordinates, by $X:Y:Z=0:\nu:-\mu$. Hence taking x, y, z as the coordinates of ξ, the equation of the line through XL, ξ is

$$\begin{vmatrix} X, & Y, & Z \\ x, & y, & z \\ 0, & \nu, & -\mu \end{vmatrix}=0,$$

viz.

$$X(\mu y+\nu z)-x(\mu Y+\nu Z)=0;$$

and at the point where this line meets the line I, the equation whereof is

$$X+Y+Z=0,$$

we have

$$(Y+Z)(\mu y+\nu z)+x(\mu Y+\nu Z)=0;$$

that is

$$Y(\mu x+\mu y+\nu z)+Z(\nu x+\mu y+\nu z)=0.$$

Hence this line, and the line

$$Yz-Zy=0,$$

with the lines

$$Y=0,\ Z=0,$$

are the lines which pass through the point YZ and the four harmonic points, and they form therefore a harmonic pencil; or we have

$$y(\mu x+\mu y+\nu z)-z(\nu x+\mu y+\nu z)=0,$$

or, what is the same thing,

$$(\mu y-\nu z)(x+y+z)+2yz(\nu-\mu)=0,$$

as the locus of the point ξ: the locus is therefore a conic passing through the points YI, ZI, YZ.

The equations of the conics which are the loci of X, Y, Z respectively, are therefore

$$\begin{aligned}U&=(\mu y-\nu z)(x+y+z)+2yz(\nu-\mu)=0,\\ V&=(\nu z-\lambda x)(x+y+z)+2zx(\lambda-\nu)=0,\\ W&=(\lambda x-\mu y)(x+y+z)+2xy(\mu-\lambda)=0;\end{aligned}$$

and the identical equation,

$$U\lambda x+V\mu y+W\nu z=0,$$

shows that these conics have three points of intersection in common. The three equations, and a fourth one to which they give rise, may be written

$$\frac{\mu}{z}-\frac{\nu}{y}+\frac{2(\nu-\mu)}{x+y+z}=0,$$

$$\frac{\nu}{x}-\frac{\lambda}{z}+\frac{2(\lambda-\nu)}{x+y+z}=0,$$

$$\frac{\lambda}{y}-\frac{\mu}{x}+\frac{2(\mu-\lambda)}{x+y+z}=0,$$

$$\frac{\nu-\mu}{x}+\frac{\lambda-\nu}{y}+\frac{\mu-\lambda}{z}=0;$$

and each of these is the equation of a conic passing through the three cubic centres.

If two of the three centres coincide, then the conics all touch at the coincident centres. Consider the first and second conics: these intersect at the point $z=0$ $x+y+z=0$; and the line $x+y+z=2kz$, if k be properly determined, or what is the same thing, the line $x+y+z=\frac{2(\theta+\nu)}{\theta}z$, if θ is properly determined, will be a line passing through the last-mentioned point and one of the other points of intersection: k or θ will of course be determined by a cubic equation; and if this has a pair of equal roots, the conics will touch. But the equation of the line, combined with those of the two conics, gives

$$x : y : z = \frac{1}{\theta+\lambda} : \frac{1}{\theta+\mu} : \frac{1}{\theta+\nu},$$

and substituting these values in the equation of the line, we have

$$\frac{1}{\theta+\lambda} + \frac{1}{\theta+\mu} + \frac{1}{\theta+\nu} - \frac{2}{\theta} = 0,$$

which is (as it should be) a cubic equation in θ.

If the equation in θ has equal roots, then

$$\frac{1}{(\theta+\lambda)^2} + \frac{1}{(\theta+\mu)^2} + \frac{1}{(\theta+\nu)^2} - \frac{2}{\theta^2} = 0;$$

and putting in these two equations,

$$x = \frac{m}{\theta+\lambda}, \quad y = \frac{m}{\theta+\mu}, \quad z = \frac{m}{\theta+\nu},$$

we have

$$x + y + z - \frac{2m}{\theta} = 0,$$

$$x^2 + y^2 + z^2 - \frac{2m^2}{\theta^2} = 0;$$

whence eliminating m,

$$(x+y+z)^2 = 2(x^2+y^2+z^2);$$

that is

$$x^2 + y^2 + z^2 - 2yz - 2zx - 2xy = 0;$$

or, what is the same thing,

$$\sqrt{x} + \sqrt{y} + \sqrt{z} = 0,$$

for the equation of the locus of the coincident centres: such locus is therefore a conic touching the lines $x=0$, $y=0$, $z=0$, in the points of intersection with the lines $y-z=0$, $z-x=0$, $x-y=0$ respectively; it is a conic touching the lines X, Y, Z harmonically in regard to the line I.

To find the envelope of the line L, the most convenient course is to take the equation in θ in the reduced form

$$\theta^3 - \theta(\mu\nu + \nu\lambda + \lambda\mu) - 2\lambda\mu\nu = 0;$$

this will have a pair of equal roots if

$$(\mu\nu + \nu\lambda + \lambda\mu)^3 - 27\lambda^2\mu^2\nu^2 = 0;$$

that is, if

$$\mu\nu + \nu\lambda + \lambda\mu - 3(\lambda\mu\nu)^{\frac{2}{3}} = 0;$$

or if

$$\frac{1}{\lambda} + \frac{1}{\mu} + \frac{1}{\nu} - 3\frac{1}{(\lambda\mu\nu)^{\frac{1}{3}}} = 0;$$

or finally if

$$\lambda^{-\frac{1}{3}} + \mu^{-\frac{1}{3}} + \nu^{-\frac{1}{3}} = 0,$$

which is the relation between λ, μ, ν in order that the line

$$\lambda x + \mu y + \nu z = 0$$

may have two coincident centres; this gives at once for the equation of the envelope

$$\sqrt[4]{x} + \sqrt[4]{y} + \sqrt[4]{z} = 0,$$

which is the equation of a curve of the fourth order having four-pointic contact with the lines $x = 0$, $y = 0$, $z = 0$, at the points of intersection with the lines $y - z = 0$, $z - x = 0$, $x - y = 0$ respectively, i.e. it has four-pointic contact with the lines X, Y, Z harmonically in regard to the line I.

It may be noticed that the rationalized form of the equation $\sqrt[4]{x} + \sqrt[4]{y} + \sqrt[4]{z} = 0$ is

$$x^4 + y^4 + z^4 - 4(yz^3 + y^3z + zx^3 + z^3x + xy^3 + x^3y) + 6(y^2z^2 + z^2x^2 + x^2y^2) - 124(x^2yz + y^2zx + z^2xy) = 0.$$

If, to fix the ideas, the signs of the coordinates x, y, z are so determined that a point *within* the triangle $x = 0$, $y = 0$, $z = 0$ has its coordinates positive (in which case the line $x + y + z = 0$ will cut the three sides *produced*), the curve $\sqrt[4]{x} + \sqrt[4]{y} + \sqrt[4]{z} = 0$ will lie wholly within the triangle, and will be of the form shown by the annexed

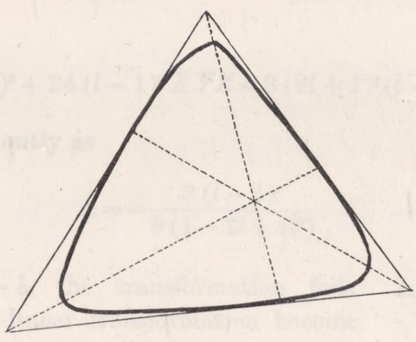

figure. This is, in fact, the form of the curve in the case considered by Plücker, where the line I is at infinity, the points of contact being the middle points of the sides. And his five groups of curves, α, β, γ, δ, ϵ, and two subdivisions of the group

β (see pp. 221—224), correspond to the following positions of the line in regard to the triangle and curve, viz.

α. The line cuts the three sides produced.

β. It passes through an angle, (a) cutting, or (b) not cutting the curve.

γ. It cuts two sides and a side produced, but does not cut or touch the curve.

δ. It cuts two sides and a side produced, and touches the curve.

ϵ. It cuts two sides and a side produced, and cuts the curve.

It is hardly necessary to remark that, in the general case, the tangential equation of the curve is

$$\xi^{-\frac{1}{3}} + \eta^{-\frac{1}{3}} + \zeta^{-\frac{1}{3}} = 0\,;$$

or what is the same thing,

$$(\eta\zeta + \zeta\xi + \xi\eta)^3 - 27\xi^2\eta^2\zeta^2 = 0\,;$$

and that the curve is therefore of the sixth class.

2, *Stone Buildings, W.C., October* 16, 1860.