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NOTE ON THE STANDARD SOLUTIONS OF A SYSTEM OF LINEAR EQUATIONS.

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To fix the ideas, the equations are assumed to be without constant terms. Supposing the system to be insufficient for the determination of the ratios of the unknown quantities, then regarding the unknown quantities as having a definite order of arrangement, there are certain solutions which may be regarded as standard solutions. Take the unknown quantities to be A, B, C, D, E, F, G, &c.; then assuming $A=0$, or else A, B, each $=0$, or else A, B, C, each $=0$, as many equations as possible, the system as thus modified will have a definite solution; for instance, the assumed equations A, B, C, D, E, each $=0$, may give a definite solution in which F is not $=0$, and it may then for convenience be put $=1$. We have thus a solution beginning with $F=1$. This being so, there will be a solution or solutions with $F=0$; we cannot then have A, B, C, D, E, each $=0$, but we again assume $A=0$, or else A, B, each $=0$, as many equations as possible; suppose A, B, C, each $=0$ give a definite solution, with D not $=0$; and then taking it for convenience to be $=1$, we have a solution beginning $D=1$, and for which $F=0$. Going on in this manner we obtain, it may be, a solution beginning $B=1$, and for which $D=0$, $F=0$; and so on, the process stopping, if not sooner, with a solution beginning $A=1$, and with the initial letters of the preceding solutions, each $=0$. We have in this manner the system of standard solutions, of a form such as

	A	B	C	D	E	F	G ...
=	0	0	0	0	0	1	*
=	0	0	0	1	*	0	*
=	0	1	0	0	*	0	*
=	1	0	*	0	*	0	*

where the * denotes a value which is not in general $=0$, but which may in any particular case happen to be so.

For instance, let it be required for the binary quartic $(a, b, c, d, e \between x, y)^4$, to find the asyzygetic seminvariants of the degree 4 and weight 6. Assuming for the seminvariant the value in the left-hand column of the diagram, the unknown coefficients being A, B, C, D, E, F, G, this must be reduced to zero by the operation

$$a\partial_b + 2b\partial_c + 3c\partial_d + 4d\partial_e\,;$$

and we thus obtain as many equations as there are terms of the degree 4 and weight 5, as appearing by the second column

			$a\partial_b + 2b\partial_c + 3c\partial_d + 4d\partial_e$	
A	a^2ce	a^2be	$2C + 2A$	$=0,$
B	„ d^2	„ cd	$D \quad + 6B + 4A$	$=0,$
C	ab^2e	ab^2d	$3F + 2D + 4C$	$=0,$
D	„ bcd	„ bc^2	$2G + 6E + 3D$	$=0,$
E	„ c^3	a^0b^3c	$4G + 3F$	$=0;$
F	a^0b^3d			
G	„ b^2c^2			

viz. the equations are

A	B	C	D	E	F	G	
2		+ 2					= 0,
4	+ 6		+ 1				= 0,
		4	+ 2		+ 3		= 0,
			3	+ 6		+ 2	= 0,
					3	+ 4	= 0.

We have first a solution beginning $B=1$, and secondly a solution beginning $A=1$, with $B=0$: the resulting two seminvariants, say P and Q, are

		$P=$	$Q=$	I	II
A	a^2ce	0	1	1	1
B	„ d^2	1	0	− 1	0
C	ab^2e	0	− 1	− 1	− 1
D	„ bcd	− 6	− 4	+ 2	− 4
E	„ c^3	+ 4	3	− 1	+ 3
F	a^0b^3d	+ 4	4		+ 4
G	„ b^2c^2	− 3	− 3		+ 3 .

As is known, there is no irreducible solution, but only the composite forms

$$\begin{aligned} \mathrm{I} &= a\,(ace - ad^2 - b^2e + 2bcd - c^3), \\ \mathrm{II} &= (ac - b^2)\,(ae - 4bd + 3c^2), \end{aligned}$$

the developed values of which are given above: II (as a form beginning $A = 1$ and with $B = 0$) can be nothing else than, and is in fact $= Q$: and so I (as a form beginning with $A = 1$, $B = -1$) can be nothing else than, and is in fact $= Q - P$; that is, we have

$$\begin{aligned} P &= -\mathrm{I} + \mathrm{II}, \quad &\text{or} \quad \mathrm{I} &= -P + Q, \\ Q &= \phantom{-\mathrm{I} +{}} \mathrm{II}, \quad & \mathrm{II} &= \phantom{-P +{}} Q; \end{aligned}$$

and so, in general, we have a standard set of values for the asyzygetic seminvariants of a given degree and weight; or, what is the same thing, for the covariants of a given deg-order.