804.

ON THE ELLIPTIC-FUNCTION SOLUTION OF THE EQUATION $x^3+y^3-1=0$.

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I HAD occasion to find elliptic-function expressions for the coordinates (x, y) of a point on the cubic curve $x^3+y^3=1$. These are derivable from the formulæ given, Legendre, *Fonctions Elliptiques*, t. I. pp. 185, 186, for the reduction to elliptic integrals of the integral $R=\int \frac{dr}{(1-z^3)^{\frac{2}{3}}}$. Legendre, writing

$$z=\frac{\sqrt{4y^3-1}-\sqrt{3}}{\sqrt{4y^3-1}+\sqrt{3}},$$

and then

$$m^3=2 \text{ and } m^2y=1+x^2,$$

finds first

$$R=m\sqrt{3}\int\frac{dx}{\sqrt{x^4+3x^2+3}};$$

and then writing $r=\sqrt[4]{3}$, $x=\tan\frac{1}{2}\phi$, and $c^2=\frac{1}{4}(2-r^2)$, finds

$$R=\tfrac{1}{2}mr\int\frac{d\phi}{\sqrt{1-c^2\sin^2\phi}};$$

we have therefore only to write $\sin\phi=\operatorname{sn}u$, to modulus

$$c, =\tfrac{1}{2}\sqrt{2-\sqrt{3}},$$

and we thence obtain an expression for z in terms of the elliptic functions $\operatorname{sn}u$, $\operatorname{cn}u$, $\operatorname{dn}u$.

Writing x instead of z, and k for c, then

$$m=\sqrt[3]{2},\quad r=\sqrt[4]{3};\quad k=\tfrac{1}{2}\sqrt{2-r^2},\quad k'=\tfrac{1}{2}\sqrt{2+r^2}.$$

Working out the substitutions, the resulting formulæ are

$$x = \frac{2r \operatorname{sn} u \operatorname{dn} u - (1 + \operatorname{cn} u)^2}{2r \operatorname{sn} u \operatorname{dn} u + (1 + \operatorname{cn} u)^2},$$

$$y = \frac{m (1 + \operatorname{cn} u) \{1 + r^2 + (1 + r^2) \operatorname{cn} u\}}{2r \operatorname{sn} u \operatorname{dn} u + (1 + \operatorname{cn} u)^2},$$

where the modulus is k as above; and these values give

$$x^3 + y^3 = 1,$$

$$\frac{dx}{(1 - x^3)^{\frac{2}{3}}}, \quad = \frac{-dy}{(1 - y^3)^{\frac{2}{3}}} = \tfrac{1}{2} mr\, du.$$

The verification is interesting enough; starting from the expression for x, and for shortness representing it by

$$x = \frac{A - B}{A + B},$$

we have

$$1 - x^3 = \frac{2B(3A^2 + B^2)}{(A + B)^3}, \quad = \frac{m^3 (1 + \operatorname{cn} u)^2 (3A^2 + B^2)}{\{2r \operatorname{sn} u \operatorname{dn} u + (1 + \operatorname{cn} u)^2\}^3}.$$

We find

$$3A^2 + B^2 = 12r^2 \operatorname{cn}^2 u \operatorname{dn}^2 u + (1 + \operatorname{cn} u)^4,$$

$$= (1 + \operatorname{cn} u) \{12r^2 (1 - \operatorname{cn} u)(k'^2 + k^2 \operatorname{cn}^2 u) + (1 + \operatorname{cn} u)^3\},$$

where the term in $\{\ \}$ is a perfect cube

$$= [1 + \operatorname{cn} u + r^2 (1 - \operatorname{cn} u)]^3.$$

The last-mentioned expression is, in fact,

$$= (1 + \operatorname{cn} u)^3 + r^2 (1 - \operatorname{cn} u) [3 (1 + \operatorname{cn} u)^2 + 3r^2 (1 + \operatorname{cn} u)(1 - \operatorname{cn} u) + r^4 (1 - \operatorname{cn} u)^2],$$

where the second term is

$$= 12r^2 (1 - \operatorname{cn} u) [\tfrac{1}{2} (1 + \operatorname{cn}^2 u) + \tfrac{1}{4} r^2 (1 - \operatorname{cn}^2 u)],$$

that is, it is

$$= 12r^2 (1 - \operatorname{cn} u)(k'^2 + k^2 \operatorname{cn}^2 u)).$$

We have consequently

$$1 - x^3 = \frac{m^3 (1 + \operatorname{cn} u)^3 \{1 + r^2 + (1 - r^2) \operatorname{cn} u\}^3}{\{2r \operatorname{sn} u \operatorname{dn} u + (1 + \operatorname{cn} u)^2\}^3},$$

or extracting the cube root $y, = \sqrt[3]{1 - x^3}$, has its foregoing value: and the differential expressions are then verified.

Suppose $y = 1$, we have

$$(m - 1)(1 + \operatorname{cn} u)^2 + mr^2 (1 - \operatorname{cn}^2 u) = 2r \operatorname{sn} u \operatorname{dn} u,$$

that is,

$$(m-1)^2(1+\text{cn}\,u)^3+2m(m-1)r^2(1+\text{cn}\,u)^2(1-\text{cn}\,u)$$
$$+3m^2(1+\text{cn}\,u)(1-\text{cn}\,u)^2=r^2(1-\text{cn}\,u)\{4-4k^2(1-\text{cn}^2\,u)\},$$

or observing that the right-hand side is

$$=r^2(1-\text{cn}\,u)\{(1+\text{cn}\,u)^2+(1-\text{cn}\,u)^2+r^2(1+\text{cn}\,u)(1-\text{cn}\,u)\},$$

and multiplying by $\frac{1}{3}r^2$, the equation becomes

$$0=\tfrac{1}{3}(m-1)^2r^2(1+\text{cn}\,u)^3+(2m^2-2m+1)(1+\text{cn}\,u)^2(1-\text{cn}\,u)$$
$$+(m^2-1)r^2(1+\text{cn}\,u)(1-\text{cn}\,u)^2-(1-\text{cn}\,u)^3;$$

viz. this is

$$0=\{\tfrac{1}{3}r^2(m^2-1)(1+\text{cn}\,u)-(1-\text{cn}\,u)\}^3,$$

as is immediately verified: hence writing $\frac{1}{3}r^2=\frac{1}{r^2}$, we have for the value in question, $y=1$,

$$(m^2-1)(1+\text{cn}\,u)-r^2(1-\text{cn}\,u)=0,$$

or say

$$m^2(1+\text{cn}\,u)=(1+\text{cn}\,u)+r^2(1-\text{cn}\,u),$$

that is,

$$\text{cn}\,u=\frac{r^2+1-m^2}{r^2-1+m^2},$$

which is one of the values of cn u derived from the equation $x=0$; but this equation $x=0$ gives, not the foregoing equation, but

$$m^6(1+\text{cn}\,u)^3=\{(1+\text{cn}\,u)+r^2(1-\text{cn}\,u)\}^3,$$

viz. the three values of cn u are the foregoing value and the two values obtained therefrom by changing m into ωm and $\omega^2 m$ respectively, ω being an imaginary cube root of unity. In fact, the curve $x^3+y^3=1$ has at the point $x=0$, $y=1$ an inflexion, the tangent being $y=1$, so that this line meets the curve in the point counting three times; but the line $x=0$ meets the curve in the point, and besides in two imaginary points.