805.

NOTE ON ABEL'S THEOREM.

[From the *Proceedings of the Cambridge Philosophical Society*, vol. IV. (1883), pp. 119—122.]

CONSIDERING Abel's theorem in so far as it relates to the first kind of integrals, and as a differential instead of an integral theorem, the theorem may be stated as follows:

We have a fixed curve $f(x, y, 1) = 0$ of the order m; this implies a relation $f'(x)\,dx + f'(y)\,dy = 0$, between the differentials dx, dy of the coordinates of a point on the curve; and we may therefore write

$$d\omega = \frac{dx}{f'(y)} = -\frac{dy}{f'(x)},$$

and, instead of dx or dy, use $d\omega$ to denote the displacement of a point (x, y) on the curve.

Taking for greater simplicity the fixed curve to be a curve without nodes or cusps, and therefore of the deficiency $\frac{1}{2}(m-1)(m-2)$, we consider its mn intersections by a variable curve $\phi(x, y, 1) = 0$ of the order n. And then, if $(x, y, 1)^{m-3}$ denote an arbitrary rational and integral function of (x, y) of the order $m-3$, the theorem is that we have between the displacements $d\omega_1, d\omega_2, \ldots, d\omega_{mn}$ of the mn points of intersection, the relation

$$\Sigma\,(x, y, 1)^{m-3}\,d\omega = 0,$$

where the left-hand side is the sum of the values of $(x, y, 1)^{m-3}\,d\omega$, belonging to the mn points of intersection respectively.

For the proof, observe that, varying in any manner the curve ϕ, we obtain

$$\frac{d\phi}{dx}\,dx + \frac{d\phi}{dy}\,dy + \delta\phi = 0,$$

where $\delta\phi$ is that part which depends on the variation of the coefficients, of the whole variation of ϕ; viz. if $\phi = ax^n + bx^{n-1}y + \ldots$, then $\delta\phi = x^n da + x^{n-1}y\, db + \ldots$; $\delta\phi$ is thus, in regard to the coordinates (x, y), a rational and integral function of the order n. Writing in this equation

$$dx,\ dy = \frac{df}{dy}\,d\omega,\ -\frac{df}{dx}\,d\omega,$$

the equation becomes

$$\left(\frac{d\phi}{dx}\frac{df}{dy} - \frac{d\phi}{dy}\frac{df}{dx}\right)d\omega + \delta\phi = 0,$$

or say

$$-J(f,\ \phi)\,d\omega + \delta\phi = 0,$$

that is,

$$d\omega = \frac{\delta\phi}{J(f,\ \phi)};$$

and then multiplying each side by the arbitrary function $(x, y, 1)^{m-3}$, we have

$$\Sigma\,(x,\ y,\ 1)^{m-3}\,d\omega = \Sigma\,\frac{(x,\ y,\ 1)^{m-3}}{J(f,\ \phi)}\,\delta\phi,$$

where $\delta\phi$ being of the order n in the variables, the numerator is a rational and integral function of (x, y) of the order $m+n-3$: hence by a theorem contained in Jacobi's paper "Theoremata nova algebraica circa systema duarum æquationum inter duas variabiles," *Crelle*, t. XIV. (1835), pp. 281—288, [*Ges. Werke*, t. III., pp. 285—294], the sum on the right-hand side is $=0$: hence the required result $\Sigma\,(x,\ y,\ 1)^{m-3}\,d\omega = 0$.

Observing that $(x, y, 1)^{m-3}$ is an arbitrary function, the equation just obtained breaks up into the equations

$$\Sigma\,d\omega = 0,\quad \Sigma x\,d\omega = 0,\quad \Sigma y\,d\omega = 0, \ldots,\quad \Sigma x^{m-3}\,d\omega = 0, \ldots,\quad \Sigma y^{m-3}\,d\omega = 0,$$

viz. the number of equations is

$$1 + 2 + \ldots + (m-2),\quad = \tfrac{1}{2}(m-1)(m-2),$$

which is $=p$, the deficiency of the curve.

Suppose the fixed curve $f(x, y, 1) = 0$ is a cubic, $m = 3$, and we have the single relation $\Sigma\,d\omega = 0$, where the summation refers to the $3n$ points of intersection of the cubic and of the variable curve of the order n, $\phi(x, y, 1) = 0$.

In particular, if this curve be a line, $n = 1$, and the equation is $d\omega_1 + d\omega_2 + d\omega_3 = 0$; here the two points (x_1, y_1), (x_2, y_2) taken at pleasure on the cubic, determine the line, and they consequently determine uniquely the third point of intersection (x_3, y_3); there should thus be a single equation giving the displacement $d\omega_3$ in terms of the displacements $d\omega_1$, $d\omega_2$; viz. this is the equation just found

$$d\omega_1 + d\omega_2 + d\omega_3 = 0.$$

So if the variable curve be a conic, $n=2$; and we have between the displacements of the six points the relation

$$d\omega_1 + d\omega_2 + \ldots + d\omega_6 = 0:$$

here five of the points determine the conic, and they therefore determine uniquely the sixth point; and there should be between the displacements a single relation as just found.

If the variable curve be a cubic, $n=3$, and we have between the displacements of the nine points the relation

$$d\omega_1 + d\omega_2 + \ldots + d\omega_9 = 0:$$

here eight of the points do *not* determine the cubic ϕ, but they nevertheless determine the ninth point, viz. (reproducing the reasoning which establishes this well-known and fundamental theorem as to cubic curves) if $\phi_0 = 0$ be a particular cubic through the 8 points, then the general cubic is $\phi_0 + kf = 0$, and the intersections with $f=0$ are given by the equations $\phi_0 = 0$, $f=0$; whence the ninth point is independent of k, and is determined uniquely by the 8 points. There should thus be a single relation between the displacements, viz. this is the relation just found.

And so if the variable curve be a quartic, or curve of any higher order, it appears in like manner that there should be a single relation between the displacements; this relation being in fact the foregoing relation $\Sigma d\omega = 0$.

But take the fixed curve to be a quartic, $m=4$: then we have between the displacements $d\omega$ the relation

$$\Sigma\,(x,\ y,\ 1)\,d\omega = 0,$$

that is, the three equations

$$\Sigma x\, d\omega = 0,\quad \Sigma y\, d\omega = 0,\quad \Sigma\, d\omega = 0.$$

If the variable curve is a conic, $n=2$, then there are 8 points of intersection; 5 of these taken at pleasure determine the conic, and they consequently determine the remaining 3 points of intersection: hence there should be 3 equations. And so if the variable curve be a curve of any higher order, then by considerations similar to those made use of in the case where the first curve is a cubic it appears that the number of equations between the displacements $d\omega$ should always be $=3$.

But if the variable curve be a line, $n=1$, then the number of the points of intersection is $=4$: 2 of these taken at pleasure determine the line, and they consequently determine the remaining 2 points of intersection; and the number of equations between the displacements $d\omega$ should thus be $=2$. But by what precedes, we have the 3 equations

$$\begin{aligned} d\omega_1 + \quad d\omega_2 + \quad d\omega_3 + \quad d\omega_4 &= 0,\\ x_1 d\omega_1 + x_2 d\omega_2 + x_3 d\omega_3 + x_4 d\omega_4 &= 0,\\ y_1 d\omega_1 + y_2 d\omega_2 + y_3 d\omega_3 + y_4 d\omega_4 &= 0; \end{aligned}$$

here the 4 points of intersection are on a line $y = ax + b$; we have therefore $y_1 = ax_1 + b, \ldots, y_4 = ax_4 + b$; the equations between the $d\omega$'s give

$$(y_1 - ax_1 - b)\, d\omega_1 + \ldots + (y_4 - ax_4 - b)\, d\omega_4 = 0,$$

that is, is a single relation $0 = 0$; or the 3 equations thus reduce themselves to 2 independent equations.

Again, if the fixed curve be a quintic, $m = 5$, there are here between the displacements the 6 equations

$$\Sigma x^2\, d\omega = 0, \quad \Sigma xy\, d\omega = 0, \quad \Sigma y^2\, d\omega = 0,$$
$$\Sigma x\, d\omega = 0, \quad \Sigma y\, d\omega = 0, \quad \Sigma\, d\omega = 0;$$

the two cases in which the number of independent equations is less than 6 are (i) when the variable curve is a line, and (ii) when the variable curve is a conic. For the line $n = 1$, and the number should be $= 3$. We have the above 6 equations; but the equation of the line is $ax + by + c = 0$, that is, we have $ax_1 + by_1 + c = 0$, &c.; we deduce the 3 identical equations

$$\Sigma x\,(ax + by + c) = 0, \quad \Sigma y\,(ax + by + c) = 0, \quad \Sigma\,(ax + by + c) = 0,$$

and the number of independent equations is thus $6 - 3$, $= 3$ as it should be.

So when the variable curve is a conic, $n = 2$; the number of independent equations should be $= 5$. The points of intersection lie on a conic $(a, b, c, f, g, h \between x, y, 1)^2 = 0$; we have therefore the several equations $(a, b, c, f, g, h \between x_1, y_1, 1)^2 = 0$, &c.: we have therefore the single identical equation

$$\Sigma\,(a, b, c, f, g, h \between x, y, 1)^2\, d\omega = 0,$$

and the number of independent equations is $6 - 1$, $= 5$ as it should be.

Obviously the like considerations apply to the case where the fixed curve is a curve of any given order whatever.