808.

NOTE ON A FORM OF THE MODULAR EQUATION IN THE TRANSFORMATION OF THE THIRD ORDER.

[From the *Messenger of Mathematics*, vol. XII. (1883), pp. 173, 174.]

IN my *Treatise on Elliptic Functions*, pp. 214—216, writing only $\frac{1}{J}$, $\frac{1}{J'}$ instead of Ω, Ω', and α, β instead of α', β', I have shown as follows: viz. if k, λ denote as usual the original modulus, and the transformed modulus, and if

$$J=\frac{(k^4+14k^2+1)^3}{108k^2(1-k^2)^4},\quad J'=\frac{(\lambda^4+14\lambda^2+1)^3}{108\lambda^2(1-\lambda^2)^4},$$

then the relation between J and J' can be found by the elimination of α, β from the equations

$$\alpha+\beta=1,$$

$$J=\frac{(1+8\alpha)^3}{64\alpha(1-\alpha)^3},\quad J'=\frac{(1+8\beta)^3}{64\beta(1-\beta)^3}.$$

By a very slight change we obtain the result given by Prof. Klein in his paper, "Ueber die Transformation der elliptischen Functionen, &c.," *Math. Ann.* t. XIV. (1879), pp. 111—172; viz., see p. 143, the relation is to be obtained by the elimination of τ, τ' from the equation $\tau\tau'=1$, and the equations

$$J\ :\ J-1\ :\ 1=(\tau-1)(9\tau-1)^3\ :\ (27\tau^2-18\tau-1)^2\ :\ -64\tau;$$
$$J'\ :\ J'-1\ :\ 1=(\tau'-1)(9\tau'-1)^3\ :\ (27\tau'^2-18\tau'-1)^2\ :\ -64\tau';$$

these last equations being equivalent to two equations only in virtue of the identity

$$(\tau-1)(9\tau-1)^3+64\tau=(27\tau^2-18\tau-1)^2,$$

and the like identity in τ'.

In fact, writing $\alpha=\frac{\tau}{\tau-1}$, $\beta=\frac{\tau'}{\tau'-1}$, the equation $\alpha+\beta=1$ becomes $\tau\tau'=1$; and then for α, β substituting their values, we have

$$J=\frac{(9\tau-1)^3(\tau-1)}{-64\tau},\quad J'=\frac{(9\tau'-1)^3(\tau'-1)}{-64\tau'},$$

which are the formulæ in question.