810.

NOTE ON A SYSTEM OF EQUATIONS.

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THE equations are

$$x^2 = ax + by, \quad xy = cx + dy, \quad y^2 = ex + fy,$$

where

$$\frac{b}{d} = \frac{a-d}{c-f} = \frac{c}{e};$$

or, what is the same thing, if a, b, c, d are given, then

$$e = \frac{cd}{b}, \quad f = c - \frac{d(a-d)}{b};$$

and this being so, the equations are equivalent to two independent equations; viz. starting from the first and the second equations, we have

$$dx^2 - bxy = (ad - bc)\,x,$$

that is,

$$dx - by = (ad - bc):$$

and thence

$$dxy - by^2 = (ad - bc)\,y,$$

or

$$d(cx + dy) - by^2 = (ad - bc)\,y;$$

which, attending to the values of e and f, is the third equation

$$y^2 = ex + fy.$$

We have

$$\frac{x}{y} = \frac{ax + by}{cx + dy}, \quad \frac{y}{x} = \frac{ex + fy}{cx + dy};$$

that is,

$$cx^2 - (a-d)\,xy - by^2 = 0,$$
$$ex^2 - (c-f)\,xy - dy^2 = 0,$$

and eliminating the (x, y) from these equations we have an equation $\Omega = 0$ as the condition that the original three equations may have a single common root; the before-mentioned equations $\frac{b}{d} = \frac{a-d}{c-f} = \frac{c}{e}$, are the conditions in order that the three equations may have two common roots, that is, that there may be *two* systems of (x, y) satisfying the three equations.

We have, moreover, $y(x-d) = cx$, $x(y-c) = dy$, and substituting these values, say $y = \frac{cx}{x-d}$ and $x = \frac{dy}{y-c}$, in the first and third equations respectively, they become

$$x - a = \frac{bc}{x-d},\quad y - f = \frac{de}{y-c},$$

that is,

$$x^2 - (a+d)\,x + ad - bc = 0,$$
$$y^2 - (c+f)\,y + cf - de = 0,$$

which are quadric equations for x and y respectively; it is easy to express the second equation (like the first) in terms of (a, b, c, d), and the first equation (like the second) in terms of (c, d, e, f), but the forms are less simple.

Suppose $(a, b, c, d) = (-1, -1, 1, 0)$, then we have $(e, f) = (0, 1)$, the two equations in $x : y$ become $x^2 + xy + y^2 = 0$, and $0 = 0$ respectively; those in x and y become $x^2 + x + 1 = 0$, $y^2 - 2y + 1 = 0$ respectively; this is right, for the three equations are

$$x^2 = -x - y,\quad xy = x,\quad y^2 = y;$$

viz. from the third equation we have $y = 1$, a value satisfying the second equation, and then the first equation becomes $x^2 + x + 1 = 0$; or, if we please, $x^2 + xy + y^2 = 0$, the values in fact being $x = \omega$, an imaginary cube root of unity, and $y = 1$.

In the general case, the values (x, y) may be regarded as units in a complex numerical theory, viz. if (a, b, c, d, e, f) are integers, and p, q, p', q', P, Q are also integers, then the product of the two complex integers $px + qy$ and $p'x + q'y$ will be a complex integer $Px + Qy$.