811.

ON THE LINEAR TRANSFORMATION OF THE THETA FUNCTIONS.

[From the *Messenger of Mathematics*, vol. XIII. (1884), pp. 54—60.]

THE functions referred to are the single Theta Functions; these may be defined as doubly infinite products, as was in fact done in my "Mémoire sur les fonctions doublement périodiques," *Liouv.* t. x. (1845), pp. 385—420, [25]; and it is interesting to consider from this point of view the theory of their linear transformation: this I propose to do in the present paper, adopting throughout the notation of Smith's* "Memoir on the Theta and Omega Functions."

The periods K, iK' are, in general, imaginary quantities

$$K = A + Bi,$$
$$iK' = C + Di,$$

where $AD - BC$ is positive; writing then $\omega = \frac{iK'}{K}$, and $q = e^{i\pi\omega}$, also for shortness

$$(q1) = 2q^{\frac{1}{4}}\Pi_1^\infty (1 - q^{2n})^3,$$

where $q^{\frac{1}{4}}$ denotes $e^{\frac{1}{4}i\pi\omega}$, the expression of the odd theta-function $\vartheta_1(x, \omega)$ as a doubly infinite product is

$$\vartheta_1(x, \omega) = (q1)\, x\Pi\Pi\left(1 + \frac{x}{m\pi + n\omega\pi}\right), \quad \left(\frac{\mu}{\nu} = \infty\right),$$

where (m, n) have any positive or negative integer values (the combination $m = 0$, $n = 0$ excluded) from $m = -\mu$ to μ, and $n = -\nu$ to ν, μ and ν being each ultimately infinite but so that μ is infinite in comparison with ν; this condition in regard to the limits is indicated by $\mu/\nu = \infty$; and similarly $\nu/\mu = \infty$ would indicate that ν was infinite in comparison with μ.

[* Smith's *Collected Mathematical Papers*, vol. II., pp. 415—621.]

The condition as to the limits might be that (m, n) have any positive or negative values (excluding as before) such that the modulus of $m+n\omega$ does not exceed a positive value T, which is ultimately taken to be infinite; this condition may be indicated by $\text{mod}=\infty$.

The values of the double product corresponding to the different conditions as to the limits are not equal, but they differ only by an exponential factor, the exponent being a multiple of x^2; we thus have

$$x\Pi\Pi\left(1+\frac{x}{mK+niK'}\right)\left(\frac{\mu}{\nu}=\infty\right)=\exp\left(\nabla x^2\right)x\Pi\Pi\left(1+\frac{x}{mK+niK'}\right)(\text{mod}=\infty),$$

where ∇ is a determinate value, depending on K and K'; and similarly

$$x\Pi\Pi\left(1+\frac{x}{m\Lambda+ni\Lambda'}\right)\left(\frac{\mu}{\nu}=\infty\right)=\exp\left(\Box x^2\right)x\Pi\Pi\left(1+\frac{x}{m\Lambda+ni\Lambda'}\right)(\text{mod}=\infty),$$

where $\Box$ is a determinate value depending in like manner on Λ, Λ'.

We have, then, as above

$$\begin{aligned}\vartheta_1(x,\omega)&=(q1)\,x\Pi\Pi\left(1+\frac{x}{m\pi+n\omega\pi}\right)\left(\frac{\mu}{\nu}=\infty\right)\\&=(q1)\frac{\pi}{K}\frac{Kx}{\pi}\Pi\Pi\left(1+\frac{\frac{Kx}{\pi}}{mK+niK'}\right)\\&=(q1)\frac{\pi}{K}\exp\left(\nabla\frac{K^2x^2}{\pi^2}\right)\frac{Kx}{\pi}\Pi\Pi\left(1+\frac{\frac{Kx}{\pi}}{mK+niK'}\right)(\text{mod}=\infty),\end{aligned}$$

viz. we have thus defined $\vartheta_1(x,\omega)$ as a doubly infinite product with the limiting condition $(\text{mod}=\infty)$; if for x we write $\frac{\pi x}{h}$, h arbitrary, we have

$$\vartheta_1\left(\frac{\pi x}{h},\omega\right)=(q1)\frac{\pi}{K}\exp\left(\nabla\frac{K^2x^2}{h^2}\right)\frac{Kx}{h}\Pi\Pi\left(1+\frac{\frac{Kx}{h}}{mK+niK'}\right)(\text{mod}=\infty),$$

and similarly, if $\Omega=\frac{i\Lambda'}{\Lambda}$, $Q=e^{i\pi\Omega}$, then

$$\begin{aligned}\vartheta_1\left\{(a+b\Omega)\frac{\pi x}{h},\ \Omega\right\}&=(Q1)\frac{\pi}{\Lambda}\exp\left\{(a+b\Omega)^2\,\Box\,\frac{\Lambda^2x^2}{h^2}\right\}\\&\qquad\times(a+b\Omega)\frac{\Lambda x}{h}\Pi\Pi\left(1+\frac{(a+b\Omega)\frac{\Lambda x}{h}}{m\Lambda+ni\Lambda'}\right)(\text{mod}=\infty).\end{aligned}$$

In the case of a linear transformation, we have

$$\omega=\begin{vmatrix}a, & b\\ c, & d\end{vmatrix}\times\Omega,\ \text{ that is, }\ \omega=\frac{c+d\Omega}{a+b\Omega},$$

where a, b, c, d are positive or negative integers such that $ad-bc=+1$; it is to be shown that the two infinite products are in this case identical; this being so, we have

$$\frac{\vartheta_1\left\{(a+b\Omega)\frac{\pi x}{h},\ \Omega\right\}}{\vartheta_1\left\{\frac{\pi x}{h},\ \omega\right\}}=\frac{(Q1)}{(q1)}\exp\left\{\{(a+b\Omega)^2\,\square\,\Lambda^2-\nabla K^2\}\frac{x^2}{h^2}\right\},$$

viz. the two functions differ only by a constant factor and by an exponential factor, the exponent being a multiple of x^2; after all reductions, this factor is found to be

$$=\exp\left(-i\pi b\,(a+b\Omega)\frac{x^2}{h^2}\right).$$

We have

$$\omega=\frac{c+d\Omega}{a+b\Omega},$$

or since

$$\omega=\frac{iK'}{K},\quad \Omega=\frac{i\Lambda'}{\Lambda},$$

this is

$$\frac{iK'}{K}=\frac{c\Lambda+di\Lambda'}{a\Lambda+bi\Lambda'},$$

or say

$$\frac{1}{M}K=a\Lambda+bi\Lambda',$$

$$\frac{1}{M}iK'=c\Lambda+di\Lambda',$$

either of which equations may be taken as a definition of the multiplier M. We have

$$\frac{K}{M\Lambda}=a+b\Omega,$$

$$\frac{1}{M}(mK+niK')=(am+cn)\Lambda+(bm+dn)i\Lambda'$$
$$=m'\Lambda+n'i\Lambda',$$

if

$$m'=am+cn,$$
$$n'=bm+dn.$$

Here to any integer values of $(m,\ n)$ there correspond integer values of m', n'; and conversely, in virtue of the equation $ad-bc=1$, to any integer values of m', n' there correspond integer values of m, n. The two products are

$$\Pi\Pi\left(1+\frac{\frac{Kx}{Mh}}{m'\Lambda+n'i\Lambda'}\right),\ (\text{mod}=\infty),$$

$$\Pi\Pi\left(1+\frac{\frac{(a+b\Omega)\,\Lambda x}{h}}{m\Lambda+ni\Lambda'}\right),\ (\text{mod}=\infty).$$

But, as above, we have $\frac{K}{M} = (a + b\Omega)\Lambda$: and then, observing that in the first of the two products we may for m', n' write m, n, it at once appears that the two products are identical.

The exponential factor, writing therein $(a + b\Omega)\Lambda = \frac{K}{M}$, becomes

$$\exp\left\{\left(\frac{\square}{M^2} - \nabla\right)\frac{K^2x^2}{h^2}\right\}.$$

The values of ∇, $\square$ are at once obtained by means of a formula* given in my Memoir, viz. we have

$$\nabla = -\tfrac{1}{2}(B + \beta),$$

where

$$B = \frac{\pi(\omega v + \omega' v')}{\Omega\Upsilon \bmod (\omega v' - \omega' v)},$$

$$\beta = \frac{\pi i}{\Omega\Upsilon}\frac{(\omega v' - \omega' v)}{\bmod (\omega v' - \omega' v)}.$$

Comparing with the present notation

$$\Omega = \omega + \omega' i, \quad = A + Bi = K,$$

$$\Upsilon = v + v' i, \quad = C + Di = K' i,$$

so that Ω, Υ denote K, $K'i$, and ω, ω', v, v' denote A, B, C, D respectively: $\omega v' - \omega' v$ is thus $= AD - BC$, which has been assumed to be positive; hence also $\bmod(\omega v' - \omega' v) = AD - BC$, and the formula becomes

$$\nabla = -\tfrac{1}{2}\pi\left\{\frac{AC + BD}{i(AD - BC)} + 1\right\}\frac{1}{KK'}.$$

Now writing

$$\Lambda = A_1 + B_1 i,$$

$$i\Lambda' = C_1 + D_1 i,$$

then we have

$$\frac{1}{M}(A + Bi) = a\Lambda + bi\Lambda' = a(A_1 + B_1 i) + b(C_1 + D_1 i),$$

$$\frac{1}{M}(C + Di) = c\Lambda + di\Lambda' = c(A_1 + B_1 i) + d(C_1 + D_1 i);$$

consequently, if

$$M = \rho(\cos\theta + i\sin\theta),$$

[* *Collected Mathematical Papers*, t. I., p. 164. The denominator factor $\Omega\Upsilon$ has been omitted (p. 165) by mistake.]

we have

$$\frac{1}{\rho}(\quad A\cos\theta + B\sin\theta) = aA_1 + bC_1,$$

$$\frac{1}{\rho}(-A\sin\theta + B\cos\theta) = aB_1 + bD_1,$$

$$\frac{1}{\rho}(\quad C\cos\theta + D\sin\theta) = cA_1 + dC_1,$$

$$\frac{1}{\rho}(-C\sin\theta + D\cos\theta) = cB_1 + dD_1,$$

and thence

$$\frac{1}{\rho^2}(AD-BC) = (ad-bc)(A_1D_1 - B_1C_1), \quad = (A_1D_1 - B_1C_1).$$

Hence $A_1D_1 - B_1C_1$ is positive, and we have

$$\square = -\tfrac{1}{2}\pi\left\{\frac{A_1C_1 + B_1D_1}{i(A_1D_1 - B_1C_1)} + 1\right\}\frac{1}{\Lambda\Lambda'}.$$

Take K_1 the conjugate of K, Λ_1 the conjugate of Λ, then

$$K_1 = A - Bi, \quad \Lambda_1 = A_1 - B_1 i,$$
$$iK' = C + Di, \quad i\Lambda' = C_1 + D_1 i.$$

We have

$$iK_1K' = AC + BD + i(AD - BC),$$

and therefore

$$\frac{K_1K'}{AD-BC} = \frac{AC+BD}{i(AD-BC)} + 1, \quad \nabla = \frac{-\tfrac{1}{2}\pi}{AD-BC}\,\frac{K_1}{K},$$

and similarly

$$\square = \frac{-\tfrac{1}{2}\pi}{A_1D_1 - B_1C_1}\,\frac{\Lambda_1}{\Lambda}.$$

The exponential is

$$\left(\frac{\square}{M^2} - \nabla\right)\frac{K^2x^2}{h^2};$$

and we have

$$\frac{\square}{M^2} - \nabla = \frac{-\tfrac{1}{2}\pi}{M^2(A_1D_1 - B_1C_1)}\,\frac{\Lambda_1}{\Lambda} + \frac{\tfrac{1}{2}\pi}{AD-BC}\,\frac{K_1}{K},$$

which is

$$= \frac{-\tfrac{1}{2}\pi}{M^2(A_1D_1 - B_1C_1)}\,\frac{\Lambda_1}{\Lambda} + \frac{\tfrac{1}{2}\pi}{\rho^2(A_1D_1 - B_1C_1)}\,\frac{K_1}{K},$$

$$= \frac{-\tfrac{1}{2}\pi}{A_1D_1 - B_1C_1}\left(\frac{1}{M^2}\,\frac{\Lambda_1}{\Lambda} - \frac{1}{\rho^2}\,\frac{K_1}{K}\right).$$

But $\rho/M = \cos\theta - i\sin\theta$, or calling this for a moment P, then $1/M^2 = P^2/\rho^2$, and the formula may be written

$$\frac{\square}{M^2} - \nabla = \frac{-\frac{1}{2}\pi P}{\rho^2(A_1D_1 - B_1C_1)}\left(P\frac{\Lambda_1}{\Lambda} - P^{-1}\frac{K_1}{K}\right)$$

$$= \frac{-\frac{1}{2}\pi P}{\rho^2(A_1D_1 - B_1C_1)}\{(\cos\theta - i\sin\theta)\,\Lambda_1 K - (\cos\theta + i\sin\theta)\,\Lambda K_1\}\frac{1}{K\Lambda}.$$

The term in $\{\ \}$ is

$$(\cos\theta - i\sin\theta)(A+Bi)(A_1 - B_1 i) - (\cos\theta + i\sin\theta)(A - Bi)(A_1 + B_1 i),$$
$$= 2\cos\theta\,[-(AB_1 - A_1B)\,i] - 2i\sin\theta\,(AA_1 + BB_1),$$
$$= -2i\,\{(AB_1 - A_1B)\cos\theta + (AA_1 + BB_1)\sin\theta\},$$
$$= -2i\,\{B_1(A\cos\theta + B\sin\theta) - A_1(-A\sin\theta + B\cos\theta)\},$$
$$= -2i\rho\,\{B_1(aA_1 + bC_1) - A_1(aB_1 + bD_1)\},$$
$$= +2i\rho b\,(A_1D_1 - B_1C_1).$$

Hence

$$\frac{\square}{M^2} - \nabla = \frac{-\frac{1}{2}\pi P}{\rho^2(A_1D_1 - B_1C_1)}\,2ib\rho\,(A_1D_1 - B_1C_1)\frac{1}{K\Lambda}$$

$$= -\frac{i\pi bP}{\rho}\,\frac{1}{K\Lambda} = -\frac{i\pi b}{M}\,\frac{1}{K\Lambda},$$

and the exponential thus is

$$= \exp\left(-\frac{i\pi b}{M}\,\frac{1}{K\Lambda}\,\frac{K^2x^2}{h^2}\right),\ = \exp\left(-i\pi b\,\frac{K}{M\Lambda}\,\frac{x^2}{h^2}\right);$$

or, since $\dfrac{K}{M\Lambda} = (a + b\Omega)$, this is

$$= \exp\left(-i\pi b\,(a + b\Omega)\,\frac{x^2}{h^2}\right);$$

and we have thus the required formula

$$\frac{\vartheta_1\left\{(a + b\Omega)\dfrac{\pi x}{h},\ \Omega\right\}}{\vartheta_1\left\{\qquad\dfrac{\pi x}{h},\ \omega\right\}} = \frac{(Q1)}{(q1)}(a + b\Omega)\exp\left(-i\pi b\,(a + b\Omega)\,\frac{x^2}{h^2}\right).$$