812.

ON ARCHIMEDES' THEOREM FOR THE SURFACE OF A CYLINDER.

[From the *Messenger of Mathematics*, vol. XIII. (1884), pp. 107, 108.]

THE measure of the surface of a cylinder was first obtained by Archimedes in his Treatise on the Sphere and Cylinder (Book I., Prop. XIV.), *Œuvres d'Archimède*, par F. Peyrard, 4° Paris, 1807, pp. 26—31; viz. Archimedes showed that the surface of the cylinder was equal to the area of a circle, having its radius a mean proportional between the height and the diameter of the circular base $[S = 2\pi ah, = \pi \{\sqrt{(2a \,.\, h)}\}^2]$.

The following is *in effect* his demonstration:

He considers regular polygons (with the same number of sides) inscribed in and circumscribed about a circle; and, as regards the cylinder, the prisms standing on these polygons.

Say for the circular base of the cylinder we have

$S^\times$ surface of circumscribed prism,

S „ „ cylinder,

S° „ „ inscribed prism;

and for the circle, having its radius a mean proportional between the height and the diameter of the circular base,

$B^\times$ area of circumscribed polygon,

B „ „ circle,

B° „ „ inscribed polygon,

where the four polygons referred to by $S^\times$, S°, $B^\times$, B° have all of them the same number of sides.

It is in the preceding propositions (by means of an axiom as to curve lines) shown that

$$S^\times > S > S^\circ, \quad B^\times > B > B^\circ;$$

and it is further shown that

$$S^\times = B^\times, \quad S^\circ = B^\circ.$$

It is moreover shown that, by taking the number of sides sufficiently large, the ratio $B^\times : B^\circ$, or say the fraction $B^\times/B^\circ$ (which is greater than 1) may be made less than any given quantity $1+\epsilon$.

It is then to be shown that $S = B$.

If not, then

either

$$B < S.$$

This being so, it is possible to make

$$B^\times/B^\circ < S/B,$$

that is,

$$S^\times/B^\circ < S/B,$$

or

$$S^\times/S < B^\circ/B,$$

which is absurd, since

$$S^\times/S > 1; \quad B^\circ/B < 1;$$

or else

$$B > S.$$

This being so, it is possible to make

$$B^\times/B^\circ < B/S,$$

that is,

$$B^\times/S^\circ < B/S,$$

or

$$B^\times/B < S^\circ/S,$$

which is absurd, since

$$B^\times/B > 1; \quad S^\circ/S < 1;$$

and consequently $S = B$, the theorem in question.

I take the opportunity of referring to two theorems by Archimedes, Lemmas, Prop. V. and VI., Peyrard, pp. 429—435, which relate to the contacts of circles. We have in each of them the figure which he calls the Arbelon, viz. if A, C, B are points in this order on the same straight line, then the figure consists of the three semicircles on the diameters AC, CB, and AB respectively, and the Arbelon is the space included between the three semi-circumferences.

In Prop. V., we have also the common tangent at C to the two semicircles AC, CB; this divides the Arbelon into two mixtilinear triangles (each bounded by the common tangent, one of the smaller semicircles, and a portion of the larger semicircle), and inscribing each of these a circle, the theorem is that the two inscribed circles are of equal magnitude.

In Prop. VI., the theorem is that the radii of the smaller semicircles being as 3 : 2, then the radius of the circle inscribed in the Arbelon is to the diameter of the larger semicircle as 6 to 19. But it is noticed that the demonstration would apply to any other value of the ratio; and, in fact, if the radii of the two smaller circles are as $a : b$, then the radius of the inscribed circle is to the diameter of the larger semicircle as ab to $a^2 + ab + b^2$, which is the general form of the theorem.