814.

ON DOUBLE ALGEBRA.

[From the *Proceedings of the London Mathematical Society*, vol. xv. (1884), pp. 185—197. Read April 3, 1884.]

1. I CONSIDER the Double Algebra formed with the extraordinary symbols, or "extraordinaries" x, y, which are such that

$$\begin{aligned} x^2 &= ax + by, \\ xy &= cx + dy, \\ yx &= ex + fy, \\ y^2 &= gx + hy, \end{aligned}$$

or, as these equations may also be written,

	x	y
x	$(a,\ b)$	$(c,\ d)$
y	$(e,\ f)$	$(g,\ h)$

,

where a, b, c, d, e, f, g, h are ordinary symbols, or say coefficients; all coefficients being commutative and associative *inter se* and with the extraordinaries x, y.

The system depends in the first instance on the eight parameters a, b, c, d, e, f, g, h; but we may, instead of the extraordinaries x, y, consider the new extraordinaries connected therewith by the linear relations $\xi = \alpha x + \beta y$, $\eta = \gamma x + \delta y$, where the coefficients α, β, γ, δ may be determined so as to establish between the eight parameters any four relations at pleasure (or, what is the same thing, α, β, γ, δ are what I call "apoclastic" constants): and the number of parameters is thus properly $8 - 4, = 4$.

2. The extraordinaries here considered are not in general associative; differing herein from the imaginaries of Peirce's Memoir, "Linear Associative Algebra" (1870),

reprinted in the *American Mathematical Journal*, t. IV. (1881), pp. 97—227, which, as appears by the title, refers only to associative imaginaries. I recall some definitions and results. The symbol x is said to be *idempotent* if $x^2 = x$, *nilpotent* if $x^2 = 0$; and the systems of associative symbols are expressed as much as may be by means of such idempotent and nilpotent symbols: thus the linear systems are $(a_1)\, x^2 = x$, $(b_1)\, x^2 = 0$. A double system composed of independent symbols, that is, symbols x, y each belonging to its own linear system, and moreover such that $xy = yx = 0$, is said to be "mixed"; thus the mixed double systems are

	x	y
x	x	0
y	0	y

,

	x	y
x	x	0
y	0	0

,

	x	y
x	0	0
y	0	0

.

But Peirce excludes these from consideration, attending only to the pure systems, which he finds to be

(a_2)

	x	y
x	x	y
y	y	0

,

(b_2)

	x	y
x	x	y
y	0	0

,

(c_2)

	x	y
x	y	0
y	0	0

.

To these, however, should be added the system

(d_2)

	x	y
x	x	0
y	y	0

;

see *post*, No. 19.

3. In the general theory, where the symbols are not in the first instance taken to be associative, we may of course establish between the coefficients such relations as will make the symbols associative; and the question presents itself to show how in this case the system reduces itself to one of Peirce's systems. This I considered in my note "On Associative Imaginaries," *Johns Hopkins University Circular*, No. 15 (1882), p. 211 [822]; I there obtained, as the general form of the commutative and associative system,

$$x^2 = ax + by,$$

$$xy = yx = cx + dy,$$

$$y^2 = \frac{cd}{b}\, x + \frac{d^2 + bc - ad}{b}\, y,$$

the relation of which to Peirce's system was, as I there remarked, pointed out to me by Mr C. S. Peirce: this will be considered in the sequel, Nos. 13 to 19.

4. Starting now with the general equations

$$x^2 = ax + by,$$
$$xy = cx + dy,$$
$$yx = ex + fy,$$
$$y^2 = gx + hy,$$

we may attempt to find an extraordinary $\xi, = \alpha x + \beta y$ (α, β coefficients), such that $\xi^2 = K\xi$ (K a coefficient). In general, K is not $=0$, and, when it is not $=0$, it may without loss of generality be taken to be $=1$; we have then $\xi^2 = \xi$, ξ an *idempotent* symbol. But K may be $=0$; and then $\xi^2 = 0$, ξ a *nilpotent* symbol. To include the two cases, I retain K, it being understood that, when K is not $=0$, it may be taken to be $=1$. We have

$$\begin{aligned}\xi^2 &= \alpha^2 (ax + by) + \alpha\beta\, (\overline{c+e}\, x + \overline{d+f}\, y) + \beta^2 (gx + hy) \\ &= \{a\alpha^2 + (c+e)\,\alpha\beta + g\beta^2\}\, x + \{b\alpha^2 + (d+f)\,\alpha\beta + h\beta^2\}\, y.\end{aligned}$$

Hence, when this is $= K\xi$, that is,

$$= K(\alpha x + \beta y),$$

we have

$$\frac{\alpha}{\beta} = \frac{a\alpha^2 + (c+e)\,\alpha\beta + g\beta^2}{\alpha^2 + (d+f)\,\alpha\beta + h\beta^2},$$

a cubic equation for the determination of the ratio $\alpha : \beta$; and, for any particular value of the ratio, we can in general determine the absolute magnitudes, so that

$$K, = \frac{1}{\alpha}\{a\alpha^2 + (c+e)\,\alpha\beta + g\beta^2\}, \quad = \frac{1}{\beta}\{b\alpha^2 + (d+f)\,\alpha\beta + h\beta^2\},$$

shall be $=1$. If, however, for the given value of the ratio we have

$$a\alpha^2 + (c+e)\,\alpha\beta + g\beta^2 = 0, \quad b\alpha^2 + (d+f)\,\alpha\beta + h\beta^2 = 0,$$

one of these equations, of course, implying the other, then the value of K is $=0$.

5. It follows that there are in general three idempotent symbols ξ, η, ζ, that is, extraordinaries such that $\xi^2 = \xi$, $\eta^2 = \eta$, $\zeta^2 = \zeta$. The cubic equation may, however, have two equal roots, or three equal roots, or it may vanish identically; in this last case, any linear function $\alpha x + \beta y$ is in general idempotent. But (as will be considered in detail further on) we may, instead of an idempotent symbol or symbols, have a nilpotent symbol or symbols. It might be convenient to use the term Potency for a symbol which is in general idempotent, but which may be nilpotent. Writing $\frac{\alpha}{\beta} = \frac{-y}{x}$, we obtain a cubic equation $\Omega = (x, y)^3 = 0$, where obviously the linear factors of Ω are the just-mentioned functions ξ, η, ζ; that is, we have ξ, η, ζ as the linear factors of the cubic function

$$\Omega, = gx^3 + (h - c - e)\, x^2 y + (a - d - f)\, xy^2 + by^3;$$

each such factor, except in the case where it is nilpotent, being determined so that it shall be idempotent. The cubic function of course vanishes identically if

$$g=0,\quad h-c-e=0,\quad a-d-f=0,\quad b=0.$$

6. Two extraordinaries $\xi, =\alpha x+\beta y$; $\eta, =\gamma x+\delta y$ (α, β, γ, δ coefficients); may be such that $\xi\eta=0$: this, of course, does not imply $\eta\xi=0$; we have, in fact,

$$\begin{aligned}\xi\eta &= (\alpha x+\beta y)(\gamma x+\delta y)\\ &= \alpha\gamma x^2+\alpha\delta xy+\beta\gamma yx+\beta\delta y^2\\ &= \alpha\gamma(ax+by)+\alpha\delta(cx+dy)+\beta\gamma(ex+fy)+\beta\delta(gx+hy)\\ &= (a\alpha\gamma+c\alpha\delta+e\beta\gamma+g\beta\delta)\,x+(b\alpha\gamma+d\alpha\delta+f\beta\gamma+h\beta\delta)\,y\,;\end{aligned}$$

and the required condition is satisfied if

$$a\alpha\gamma+c\alpha\delta+e\beta\gamma+g\beta\delta=0,$$
$$b\alpha\gamma+d\alpha\delta+f\beta\gamma+h\beta\delta=0.$$

Writing these equations first under the form

$$\gamma(a\alpha+e\beta)+\delta(c\alpha+g\beta)=0,$$
$$\gamma(b\alpha+f\beta)+\delta(d\alpha+h\beta)=0,$$

and then under the form

$$\alpha(a\gamma+c\delta)+\beta(e\gamma+g\delta)=0,$$
$$\alpha(b\gamma+d\delta)+\beta(f\gamma+h\delta)=0,$$

we have

$$(a\alpha+e\beta)(d\alpha+h\beta)-(b\alpha+f\beta)(c\alpha+g\beta)=0,$$

a quadric equation for the determination of $\alpha:\beta$; and then

$$(a\gamma+c\delta)(f\gamma+h\delta)-(b\gamma+d\delta)(e\gamma+g\delta)=0,$$

a quadric equation for the determination of $\gamma:\delta$; that is, there are two values of the left-hand factor ξ, and two values of the right-hand factor η. But, of course, these correspond each to each, viz. either factor being given, the other factor is determined uniquely.

Writing successively $\dfrac{\alpha}{\beta}=\dfrac{-y}{x}$, and $\dfrac{\gamma}{\delta}=\dfrac{-y}{x}$, we have the quadric functions $(x, y)^2$,

$$\Phi\ =(eh-fg)\,x^2+(-ah-de+bg+cf)\,xy+(ad-bc)\,y^2,$$
$$\Phi_1=(ch-dg)\,x^2+(-ah-cf+bg+de)\,xy+(af-be)\,y^2,$$

where the linear factors of Φ are the two values ξ_1, ξ_2 of the left-hand factor ξ, and the linear factors of Φ_1 are the two values η_1, η_2 of the right-hand factor η.

7. In the commutative case, $c=e$, and $d=f$, we have

$$\Phi=\Phi_1=(ch-dg)\,x^2+(-ah+bg)\,xy+(ad-bc)\,y^2\,;$$

here $\xi\eta=0$, $\eta\xi=0$, and the values may be taken to be $(\xi_1, \eta_1)=(\xi, \eta)$, $(\xi_2, \eta_2)=(\eta, \xi)$, so that $\xi_1\xi_2=\xi\eta, =\eta_1\eta_2$; that is, $\Phi=\Phi_1$, as above. The value of Ω is

$$\Omega = gx^3 + (h-2c)\,x^2y + (a-2d)\,xy^2 + by^3.$$

8. In the commutative and associative case, taking ξ, η, ζ to be the three idempotent symbols, $\xi^2=\xi$, $\eta^2=\eta$, $\zeta^2=\zeta$, we have

$$\xi(\eta-\xi\eta) = \xi\eta - \xi^2\eta = \xi\eta - \xi\eta = 0\,;$$

and in this manner we have the six equations

$$\xi(\eta-\xi\eta)=0,\quad \xi(\zeta-\xi\zeta)=0\,;\quad \eta(\xi-\eta\xi)=0,\quad \eta(\zeta-\eta\zeta)=0\,;\quad \zeta(\xi-\zeta\xi)=0,\quad \zeta(\eta-\zeta\eta)=0\,;$$

viz. regarding the right-hand factor as being in each case expressed as a linear function of x, y, we have apparently six products of two linear factors, each $=0$. There is only one such product $\Phi=0$; hence, disregarding coefficients, each of the six products must be $=\Phi$, or it must be identically $=0$, viz. this will be the case if the second factor be $=0$. We hence conclude that two of the symbols ξ, η, ζ, suppose ξ and η, must be factors of Φ, viz. Φ must be $=\xi\eta$. We have $\Omega=\xi\eta\zeta$; consequently $\Omega=\zeta\Phi$, that is, two of the three linear factors of Ω are the symbols ξ, η, which are such that $\xi\eta\,(=\eta\xi)=0$. To complete the theory, observe that ζ must be a linear function of $\xi, \eta, =a\xi+b\eta$ suppose (a, b coefficients, neither of them $=0$); we thence have

$$\zeta^2,\ =a^2\xi^2+b^2\eta^2,\ =a^2\xi+b^2\eta,\ =\zeta,\ =a\xi+b\eta\,;$$

that is, $(a^2-a)\,\xi+(b^2-b)\,\eta=0$; whence $a=1$, $b=1$, and therefore $\zeta=\xi+\eta$; hence also $\xi\zeta=\xi$ and $\eta\zeta=\eta$; $\zeta-\xi\zeta=\eta$, $\zeta-\eta\zeta=\xi$. The six products consequently are $\xi\eta, \xi\eta, \eta\xi, \eta\xi, \zeta0, \zeta0$, each $=\Phi$ or identically $=0$.

9. In verification of the theorem that for the commutative and associative system the cubic function Ω contains the quadric function Φ as a factor, we may write, as above,

$$g=\frac{cd}{b},\quad h=\frac{d^2+bc-ad}{b},$$

values which give

$$\begin{aligned} b\Phi &= (bc^2-acd)\,x^2+(-ad^2-abc+a^2d+bcd)\,xy+(abd-b^2c)\,y^2\\ &= -(ad-bc)\,\{cx^2+(d-a)\,xy-by^2\},\\ b\Omega &= cdx^3+(d^2-bc-ad)\,x^2y+(ab-2bd)\,xy^2+b^2y^3,\\ &= (dx-by)\,\{cx^2+(d-a)\,xy-by^2\}, \end{aligned}$$

which gives the theorem in question. And observe further that

$$\begin{aligned} (dx-by)^2 &= d^2(ax+by)-2bd\,(cx+dy)+b^2\left(\frac{cd}{b}\,x+\frac{d^2+bc-ad}{b}\,y\right),\\ &= (ad-bc)\,(dx-by)\,; \end{aligned}$$

that is, disregarding coefficients, the two idempotent symbols ξ, η are the linear factors of $cx^2+(d-a)\,xy-by^2$, and the third idempotent symbol ζ is $=dx-by$.

10. Introducing coefficients in order to make the symbols ξ, η, ζ idempotent, and writing accordingly

$$\xi = \frac{1}{K}\{cx + \tfrac{1}{2}(d - a + \sqrt{\nabla})\, y\}, \quad \nabla = (d-a)^2 + 4bc,$$

so that

$$\eta = \frac{1}{L}\{cx + \tfrac{1}{2}(d - a - \sqrt{\nabla})\, y\}, \quad \xi\eta = \frac{c}{KL}\{cx^2 + (d-a)\,xy - by^2\},$$

$$\zeta = \frac{1}{P}(dx - by),$$

we have to verify that it is possible to determine K, L, P so that $\xi^2 = \xi$, $\eta^2 = \eta$, $\zeta^2 = \zeta$, $\zeta = \xi + \eta$. The last equation gives

$$\frac{d}{P} = \frac{c}{K} + \frac{c}{L},$$

$$-\frac{2b}{P} = \frac{d - a + \sqrt{\nabla}}{K} + \frac{d - a - \sqrt{\nabla}}{L},$$

and we thence have

$$\frac{d\,(d-a) + 2bc - d\sqrt{\nabla}}{P} = -\frac{2c\sqrt{\nabla}}{K},$$

$$\frac{d\,(d-a) + 2bc + d\sqrt{\nabla}}{P} = \frac{2c\sqrt{\nabla}}{L},$$

and we can from the equation $\zeta^2 = \zeta$ find P; viz. comparing the coefficients of x, we have

$$\frac{d}{P} = \frac{1}{P^2}\left(ad^2 - 2b\,dc + b^2\frac{cd}{b}\right), \ = \frac{d}{P^2}(ad - bc), \text{ that is, } P = (ad - bc),$$

or the values of K and L are

$$\frac{\{d\,(d-a) + 2bc - d\sqrt{\nabla}\}}{ad - bc} = -\frac{2c\sqrt{\nabla}}{K},$$

$$\frac{\{d\,(d-a) + 2bc + d\sqrt{\nabla}\}}{ad - bc} = +\frac{2c\sqrt{\nabla}}{L},$$

which should agree with the values of K and L found from the equations $\xi^2 = \xi$, $\eta^2 = \eta$, respectively. Comparing the coefficients of x, the first of these equations gives

$$\frac{c}{K} = \frac{1}{K^2}\left\{c^2 a + c\,(d - a + \sqrt{\nabla})\,c + \tfrac{1}{4}(d - a + \sqrt{\nabla})^2\frac{cd}{b}\right\}$$

$$= \frac{c}{4bK^2}\left\{4abc + 4bc\,(d-a) + d\,\{(d-a)^2 + \nabla\} + 2\,\{2bc + d\,(d-a)\}\sqrt{\nabla}\right\}$$

$$= \frac{c}{2bK^2}\left\{d\nabla + \{d\,(d-a) + 2bc\}\sqrt{\nabla}\right\},$$

that is,

$$K = \frac{1}{2b}\sqrt{\nabla}\,\{d\,(d - a + 2bc) + d\sqrt{\nabla}\},$$

and the equation for K becomes

$$\frac{d(d-a)+2bc-d\sqrt{\nabla}}{ad-bc}=\frac{-4bc}{d(d-a)+2bc+d\sqrt{\nabla}};$$

that is,

$$\{d(d-a)+2bc\}^2-d^2\{(d-a)^2+4bc\}=-4bc(ad-bc),$$

which is right; and similarly the equation for L leads to this same equation.

11. We may now establish, on the principles appearing in No. 5, the different forms of the system. Using *idem* and *nil* as abbreviations for idempotent and nilpotent respectively, there are in all 11 cases.

(i) 3 idems. Taking two of these to be x and y, the system is

$$x^2=x,\quad xy=cx+dy,\quad yx=ex+fy,\quad y^2=y.$$

Hence $\Omega=(1-c-e)x^2y+(1-d-f)xy^2$, so that the third factor is

$$(1-c-e)x+(1-d-f)y.$$

This must not reduce itself to x or y, for, if so, there would be a twofold idem; viz. as negative conditions we must have $c+e\neq 1$, $d+f\neq 1$.

And we have

$$\{(1-c-e)x+(1-d-f)y\}^2=\{1-(c+e)(d+f)\}[(1-c-e)x+(1-d-f)y],$$

which must be an idem: viz. we have the further negative condition $(c+e)(d+f)\neq 1$.

(ii) 2 idems and 1 nil. This arises from (i) by assuming therein

$$(c+e)(d+f)=1,\text{ say } d+f=\frac{1}{c+e}:$$

for then, writing $z=-(c+e)x+y$, we have

$$\begin{aligned}z^2&=(c+e)^2x-(c+e)\{(c+e)x+(d+f)y\}+y,\\&=\{1-(c+e)(d+f)\}y,\ =0;\end{aligned}$$

viz. z is a nil. And, if in the equations instead of the idem y we introduce the nil z, then the equations assume the form

$$x^2=x,\quad xz=[(c+e)d-e]x+dz,\quad zx=[(c+e)f-c]x+fz,\quad z^2=0;$$

with the idem $y=(c+e)x+z$: hence the negative conditions $c+e\neq 1$ or 0, implying $d+f\neq 1$.

But the equations are obtained in a more simple form by taking x for the idem and y for the nil; viz. we then have $x^2=x$, $xy=cx+dy$, $yx=ex+fy$, $y^2=0$: we must then have z, $=-(c+e)x+(1-d-f)y$, for an idem; this gives $z^2=-(c+e)(d+f)z$, and we have the negative conditions $c+e\neq 0$, $d+f\neq 0$ or 1.

(iii) 1 idem and 2 nils. This may be deduced from (ii) by writing therein $d+f=0$; for then $z, = -(c+e)x+y$, is a nil. The equations are $x^2=x$, $xy=cx+dy$, $yx=ex-dy$, $y^2=0$: and if, instead of x, we introduce therein z by the equation $z=-(c+e)x+y$, the equations become $y^2=0$, $yz=[-e+d(c+e)]y-ez$, $zy=[-c-d(c+e)]y+cz$, $z^2=0$, with the negative condition $c+e\neq 0$.

But it is more simple to take x, y as the nils: the equations then are $x^2=0$, $xy=cx+dy$, $yx=ex+fy$, $y^2=0$. We must have $z, =(c+e)x+(d+f)y$, an idem: this gives $z^2=(c+e)(d+f)z$; and we have the negative conditions $c+e\neq 0$, $d+f\neq 0$.

(*) We cannot have three nils. For in (iii), to make z a nil, we must have $c+e=0$ or $d+f=0$, and in the two cases respectively $z, =(c+e)x+(d+f)y$, becomes $=x$ and $=y$; so that x or y is a twofold nil. Or, what comes to the same thing, we have

$$\Omega=-(c+e)x^2y-(d+f)xy^2,$$

and Ω has a twofold factor if $c+e=0$ or $d+f=0$.

(iv) A twofold idem and a onefold idem. Taking x for the twofold idem and y for the onefold idem, Ω must reduce itself to $(1-c-e)x^2y$, viz. we must have $d+f=1$, or say $f=1-d$. The equations are $x^2=x$, $xy=cx+dy$, $yx=ex+(1-d)y$, $y^2=y$; and we have the negative condition $c+e\neq 1$, for otherwise Ω would vanish identically.

(v) A twofold idem and a onefold nil. Taking x for the twofold idem and y for the onefold nil, then the equations are $x^2=x$, $xy=cx+dy$, $yx=ex+(1-d)y$, $y^2=0$; and we have the negative condition $c+e\neq 0$.

(vi) A twofold nil and a onefold idem. Taking these to be x and y, then $d+f=0$, and the equations are $x^2=0$, $xy=cx+dy$, $yx=ex-dy$, $y^2=y$; and we have the negative condition $c+e\neq 1$.

(vii) A twofold nil and a onefold nil. Taking these to be x and y, we have $d+f=0$, and the equations are $x^2=0$, $xy=cx+dy$, $yx=ex-dy$, $y^2=0$; with the negative condition $c+e\neq 0$.

(viii) A threefold idem. Taking this to be x, then Ω must reduce itself to gx^3, viz. we must have $h=c+e$, $1=d+f$; and the equations are $x^2=x$, $xy=cx+dy$, $yx=ex+(1-d)y$, $y^2=gx+(c+e)y$; we have the negative condition $g\neq 0$, for otherwise Ω would vanish identically.

(ix) A threefold nil. Taking this to be x, then we must have $h=c+e$, $0=d+f$; the equations are $x^2=0$, $xy=cx+dy$, $yx=ex-dy$, $y^2=gx+(c+e)y$; and there is again the negative condition $g\neq 0$.

(x) $\Omega=0$ identically: infinity of idems, 1 nil. Ω will vanish identically if $g=0$, $h=c+e$, $a=d+f$, $b=0$. If there is 1 idem, there will be an infinity of idems, and 1 nil. For, assume an idem x, $x^2=x$; and, if possible, let there be no other idem; then there will be a nil y, $y^2=0$. We have $c+e=0$, $d+f=1$; and the equations are $x^2=x$, $xy=cx+dy$, $yx=-cx+(1-d)y$, $y^2=0$; whence $xy+yx=y$. Taking α, β arbitrary coefficients, we have

$$(\alpha x+\beta y)^2=\alpha^2 x+\alpha\beta y, \;=\alpha(\alpha x+\beta y);$$

hence $\alpha x+\beta y$ is an idem, except in the case $\alpha=0$, when it is the original nil y.

If besides the idem x we have an idem y, then the conditions are $c+e=1$, $d+f=1$: the equations are

$$x^2=x, \quad xy=cx+dy, \quad yx=(1-c)x+(1-d)y, \quad y^2=y;$$

whence $xy+yx=x+y$. Considering the combination $\alpha x+\beta y$, we have

$$(\alpha x+\beta y)^2=\alpha^2 x+\alpha\beta(x+y)+\beta^2 y, \quad =(\alpha+\beta)(\alpha x+\beta y).$$

This is an idem, except in the case $\alpha+\beta=0$, when it is a nil; or, say we have the single nil $x-y$. We have thus again an infinity of idems, 1 nil.

(xi) $\Omega=0$ identically; an infinity of nils. Taking the two nils x and y, the conditions are $c+e=0$, $d+f=0$; the equations are $x^2=0$, $xy=cx+dy$, $yx=-cx-dy$, $y^2=0$; whence $xy+yx=0$. Considering the arbitrary combination $\alpha x+\beta y$, we have

$$(\alpha x+\beta y)^2=\alpha\beta(xy+yx), \;=0,$$

viz. $\alpha x+\beta y$ is a nil; or there are an infinity of nils.

12. The different cases may be grouped together as follows:—

A. 2 idems, (i), (ii), (iv), (x).

Equations $x^2=x$, $xy=cx+dy$, $yx=ex+fy$, $y^2=y$.

B. 1 idem and 1 nil, (ii), (iii), (v), (vi), (x).

Equations $x^2=x$, $xy=cx+dy$, $yx=ex+fy$, $y^2=0$.

C. 2 nils, (iii), (vii), (xi).

Equations $x^2=0$, $xy=cx+dy$, $yx=ex+fy$, $y^2=0$.

D. Threefold idem, (viii).

Equations $x^2=x$, $xy=cx+dy$, $yx=ex-(1-d)y$, $y^2=gx+(c+e)y$.

E. Threefold nil, (ix).

Equations $x^2=0$, $xy=cx+dy$, $yx=ex-dy$, $y^2=gx+(c+e)y$.

The several cases of A, B, C respectively are distinguished by negative conditions which need not be here repeated.

13. I consider, as in my Note before referred to, the conditions in order that the system may be associative. We have the 8 products, x^3, x^2y, xyx, xy^2, yx^2, y^2x, yxy, y^3, giving rise to equations $x.x^2=x^2.x$, $x.xy=x^2.y$, ..., $y.y^2=y^2.y$, which, on putting therein for x^2, xy, yx, y^2 their values, must be satisfied identically. We thus obtain in

the first instance 16 relations, but some of these are repeated, and we have actually only 12 relations; viz. the relations are

$$\begin{array}{ll}
\text{(twice)} & b\,(c-e)=0,\\
& b\,(f-d)=0,\\
& g\,(c-e)=0,\\
\text{(twice)} & g\,(f-d)=0,\\
\text{(twice)} & bg-cd=0,\\
\text{(twice)} & bg-ef=0,\\
& c\,(c-h)+g\,(d-a)=0,\\
& d\,(d-a)+b\,(c-h)=0,\\
& e\,(e-h)+g\,(f-a)=0,\\
& b\,(e-h)+f\,(f-a)=0,\\
& a\,(c-e)-cf+de=0,\\
& h\,(f-d)-cf+de=0.
\end{array}$$

14. From the first four equations it appears that either $b=0$ and $g=0$, or else $c=e$ and $d=f$. I attend first to the latter case, viz. we have here the commutative system

$$x^2=ax+by,\ xy=yx=cx+dy,\ y^2=gx+hy.$$

In order that this may be associative, we must still have the relations

$$bg-cd=0,$$
$$c\,(c-h)+g\,(d-a)=0,$$
$$d\,(d-a)+b\,(c-h)=0,$$

or, as they may be written,

$$\begin{vmatrix} b, & -c, & d-a \\ -d, & g, & c-h \end{vmatrix}=0.$$

These are satisfied by $g=\dfrac{cd}{b}$, $h=\dfrac{d^2+bc-ad}{b}$, and we have thus the commutative and associative system of the Note.

Every system is of the form A, B, C, D, or E; and it can be shown that the commutative and associative system is not of the form D. For, if D were commutative, we should have $e=c$, $d=\tfrac{1}{2}$, viz. the equations will be $x^2=x$, $xy=yx=cx+\tfrac{1}{2}y$, $y^2=gx+2cy$, that is,

$$a,\ b,\ c,\ d,\ g,\ h=1,\ 0,\ c,\ \tfrac{1}{2},\ g,\ 2c;$$

and the last of the three relations, viz. $d\,(d-a)+b\,(c-h)=0$, would thus be $\tfrac{1}{2}(\tfrac{1}{2}-1)=0$, which is not satisfied. Hence the commutative and associative system can only be of one of the forms A, B, C, and E.

15. First, if the form be A, B, or C; there will be the two idem-or-nil symbols x and y, that is, we may assume $b=0$, $g=0$; and the associative conditions then become $cd=0$, $c(c-h)=0$, $d(d-a)=0$, viz. for the forms A, B, C,

A. $x^2=x$, $y^2=y$,

B. $x^2=x$, $y^2=0$,

C. $x^2=0$, $y^2=0$,

these are $cd=0$, $c(c-1)=0$, $d(d-1)=0$; $c=0$ or 1, $d=0$ or 1,

" $cd=0$, $c^2=0$, $d(d-1)=0$; $c=0$, $d=0$ or 1,

" $cd=0$, $c^2=0$, $d^2=0$; $c=0$, $d=0$.

But for the form A, if $c=0$, $d=1$, that is, $xy=yx=y$, then, writing $z=x-y$, we have $z^2=z$, $yz=zy=0$, $y^2=y$. And similarly, if $c=1$, $d=0$, that is, $xy=yx=x$, then, writing $z=-x+y$, we have $z^2=z$, $zx=xz=0$, $x^2=x$. That is, each of these is reduced to the first case $c=0$, $d=0$; that is, $x^2=x$, $xy=yx=0$, $y^2=y$.

For the form B, if $c=0$, $d=1$, then the system is $x^2=x$, $xy=yx=y$, $y^2=0$; and this cannot be reduced to the first case $x^2=x$, $xy=yx=0$, $y^2=0$.

For the form C, there is only one case, as above.

For the form E, we have $a=0$, $b=0$, ($c=e$, $d=0$, in order that the system may be commutative), $h=2c$, viz. the equations must be $x^2=0$, $xy=yx=cx$, $y^2=gx+2cy$. The associative conditions then give $c=0$; or, the system is $x^2=0$, $xy=yx=0$, $y^2=gx$. Writing $\frac{x}{g}$ instead of x, and for convenience interchanging x and y, the equations are $x^2=y$, $xy=yx=0$, $y^2=0$.

16. The commutative associative system is thus seen to be reducible as follows:—

A. system is $x^2=x$, $xy=yx=0$, $y^2=y$, first mixed system, see No. 2.

B. " $x^2=x$, $xy=yx=y$, $y^2=0$, Peirce's system (a_2),

or else

B. " $x^2=x$, $xy=yx=0$, $y^2=0$, second mixed system.

C. " $x^2=0$, $xy=yx=0$, $y^2=0$, third mixed system.

E. " $x^2=y$, $xy=yx=0$, $y^2=0$, Peirce's system (c_2).

I said, at the end of my Note before referred to, that it had been pointed out to me "that my system [the commutative associative system], in the general case $ad-bc$ not $=0$, is expressible as a mixture of two algebras of the form (a_1), see *American Journal of Mathematics*, vol. IV., p. 120; whereas, if $ad-bc=0$, it is reducible to the form (c_2), see p. 122 (*l.c.*)." The accurate conclusion is as above, that the commutative associative system is either a mixed system of one of the three forms, or else a system (a_2), or (c_2).

17. Considering next the non-commutative associative systems, we have here, *ante*, No. 14, $b=0$, $g=0$; and the relations which remain to be satisfied then are

$$cd=0,\ ef=0,\ c(c-h)=0,\ d(d-a)=0,\ e(e-h)=0,\ f(f-a)=0,$$
$$a(c-e)-cf+de=0,\ h(f-d)-cf+de=0.$$

The first equation gives $cd=0$, that is, $c=0$ or $d=0$; but we may attend exclusively to the case $c=0$, for the case $d=0$ may be deduced from this by the interchange of x, y. We have then $ef=0$; and it will be convenient to separate the cases

I. $c=0,\ e=0,\ f=0$, giving $d(d-a)=0,\quad dh=0$,

II. $c=0,\ e=0,\ f\neq 0$, " $d(d-a)=0,\ f-a=0,\ h(f-d)=0$,

III. $c=0,\ e\neq 0,\ f=0$, " $d(d-a)=0,\ e-h=0,\quad (d-a)=0,\ d(e-h)=0$,

that is, $d-a=0,\ e-h=0$.

18. We have thus five cases;

I. (a) $d=0$: $x^2=ax$, $xy=yx=0$, $y^2=hy$: commutative, and so included in what precedes.

I. (b) $d=a$, $h=0$: $x^2=ax$, $xy=ay$, $yx=0$, $y^2=0$: or, writing as we may do $a=1$, this is $x^2=x$, $xy=y$, $yx=0$, $y^2=0$; which is Peirce's system (b_2).

II. (c) $d=f=a$: $x^2=ax$, $xy=yx=ay$, $y^2=hy$: commutative, and so included in what precedes.

II. (d) $d=0$, $f=a$, $h=0$: $x^2=ax$, $xy=0$, $yx=ay$, $y^2=0$; or, writing as we may do $a=1$, this is $x^2=x$, $xy=0$, $yx=y$, $y^2=0$; which is the system (d_2).

III. (e) $d=a$, $e=h$: $x^2=ax$, $xy=ay$, $yx=hx$, $y^2=hy$; or, writing as we may do $a=1$, $h=1$, this is $x^2=x$, $xy=y$; $yx=x$; $y^2=y$. Introducing here the new symbol z, $=x-y$, we have $z^2=0$, $xz=z$, $zx=0$, $yz=z$, $zy=0$. Thus x, z form the system $x^2=x$, $xz=z$, $zx=0$, $z^2=0$ (or, what is the same thing, y, z form a system $y^2=y$, $yz=z$, $zy=0$, $z^2=0$); each of these is Peirce's system (b_2).

The conclusion is that every non-commutative associative system is either Peirce's system (b_2), or else the omitted system (d_2). Hence, disregarding the mixed systems, every associative system is either (a_2), (b_2), (c_2), or (d_2).

19. It may be proper to show that the systems (b_2), $x^2=x$, $xy=y$, $yx=0$, $y^2=0$, and (d_2), $x^2=x$, $xy=0$, $yx=y$, $y^2=0$, or say

	a	b	c	d	e	f	g	h
(b_2)	1	0	0	1	0	0	0	0
(d_2)	1	0	0	0	0	1	0	0,

are really distinct from each other. Observe that they each belong to the case (x), $\Omega=0$, an infinity of idems and 1 nil; viz. in each of them writing $z=x+\beta y$, β an arbitrary coefficient, we have $z^2=x^2+\beta(xy+yx)+\beta^2y^2$, $=x+\beta y$, $=z$, we have z an idem, and y is the only nil. And, this being so, we have in the first system $zy=y$, $yz=0$, viz. the system is $z^2=z$, $zy=y$, $yz=0$, $y^2=0$, retaining, when we write z for x, its original form. And similarly, in the second system, $zy=0$, $yz=y$; viz. the system is $z^2=z$, $zy=0$, $yz=y$, $y^2=0$, retaining, when we write therein z for x, its original form. The two are thus distinct systems, in no wise transformable the one into the other.