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ON THE SIXTEEN-NODAL QUARTIC SURFACE.

[From *Crelle's Journal der Mathem.*, t. XCIV. (1883), pp. 270—272.]

RIEMANN'S theory of the bitangents of a plane quartic leads at once to a very simple form of the equation of the sixteen-nodal surface: viz. if ξ, η, ζ denote linear functions of the coordinates (x, y, z, w) such that identically

$$x + y + z + \xi + \eta + \zeta = 0,$$
$$ax + by + cz + f\xi + g\eta + h\zeta = 0,$$

(where $af = bg = ch = 1$), then the quartic surface

$$\sqrt{x\xi} + \sqrt{y\eta} + \sqrt{z\zeta} = 0$$

has the sixteen singular tangent planes (each touching it along a conic)

$$x = 0,\quad y = 0,\quad z = 0,\quad \xi = 0,\quad \eta = 0,\quad \zeta = 0,$$
$$x + y + z = 0,\quad ax + by + cz = 0,$$
$$\xi + y + z = 0,\quad f\xi + by + cz = 0,$$
$$x + \eta + z = 0,\quad ax + g\eta + cz = 0,$$
$$x + y + \zeta = 0,\quad ax + by + h\zeta = 0,$$
$$\frac{x}{1 - bc} + \frac{y}{1 - ca} + \frac{z}{1 - ab} = 0,\quad \frac{\xi}{1 - gh} + \frac{\eta}{1 - hf} + \frac{\zeta}{1 - fg} = 0:$$

and it is thus a sixteen-nodal surface.

I have formerly given the equation of this surface under the form

$$\sqrt{x(X - w)} + \sqrt{y(Y - w)} + \sqrt{z(Z - w)} = 0,$$

where

$$\begin{array}{ll} \alpha + \beta + \gamma = 0, & X = \alpha(\gamma'\gamma'' y - \beta'\beta'' z), \\ \alpha' + \beta' + \gamma' = 0, & Y = \beta(\alpha'\alpha'' z - \gamma'\gamma'' x), \\ \alpha'' + \beta'' + \gamma'' = 0, & Z = \gamma(\beta'\beta'' x - \alpha'\alpha'' y), \\ P = \dfrac{x}{\alpha} + \dfrac{y}{\beta} + \dfrac{z}{\gamma}, & X' = \alpha'(\gamma''\gamma y - \beta''\beta z), \\ P' = \dfrac{x}{\alpha'} + \dfrac{y}{\beta'} + \dfrac{z}{\gamma'}, & Y' = \beta'(\alpha''\alpha z - \gamma''\gamma x), \\ P'' = \dfrac{x}{\alpha''} + \dfrac{y}{\beta''} + \dfrac{z}{\gamma''}, & Z' = \gamma'(\beta''\beta x - \alpha''\alpha y), \\ & X'' = \alpha''(\gamma\gamma' y - \beta\beta' z), \\ & Y'' = \beta''(\alpha\alpha' z - \gamma\gamma' x), \\ & Z'' = \gamma''(\beta\beta' x - \alpha\alpha' y), \end{array}$$

and where the equations of the sixteen singular tangent planes are

$$\begin{array}{llll} x = 0, & y = 0, & z = 0, & w = 0, \\ X - w = 0, & Y - w = 0, & Z - w = 0, & P = 0, \\ X' - w = 0, & Y' - w = 0, & Z' - w = 0, & P' = 0, \\ X'' - w = 0, & Y'' - w = 0, & Z'' - w = 0, & P'' = 0; \end{array}$$

see *Crelle's Journal*, vol. LXXIII. (1871), pp. 292, 293, [442], and also *Proc. Lond. Math. Soc.*, vol. III. (1871), p. 251*, [454].

To identify the two forms, using $x', y', z', \xi', \eta', \zeta'$ for the new form, I assume

$$x', y', z', \xi', \eta', \zeta' = lx,\ my,\ nz,\ p(X-w),\ q(Y-w),\ r(Z-w),$$

where $lp = mq = nr = 1$; and so convert the equation

$$\sqrt{x(X-w)} + \sqrt{y(Y-w)} + \sqrt{z(Z-w)} = 0$$

into

$$\sqrt{x'\xi'} + \sqrt{y'\eta'} + \sqrt{z'\zeta'} = 0.$$

The constants (l, m, n, p, q, r) and (a, b, c, f, g, h), where $af = bg = ch = 1$, are then to be determined so that we may have identically

$$\begin{array}{l} x' + y' + z' + \xi' + \eta' + \zeta' = 0, \\ ax' + by' + cz' + f\xi' + g\eta' + h\zeta' = 0, \end{array}$$

and we thus obtain 8 new equations to be satisfied by the 12 constants, viz. these are

$$\begin{array}{l} l + r\,.\,\gamma\beta'\beta'' - q\,.\,\beta\gamma'\gamma'' = 0, \\ m + p\,.\,\alpha\gamma'\gamma'' - r\,.\,\gamma\alpha'\alpha'' = 0, \\ n + q\,.\,\beta\alpha'\alpha'' - p\,.\,\alpha\beta'\beta'' = 0, \\ p + q + r = 0, \\ al + hr\,.\,\gamma\beta'\beta'' - gq\,.\,\beta\gamma'\gamma'' = 0, \\ bm + fp\,.\,\alpha\gamma'\gamma'' - hr\,.\,\gamma\alpha'\alpha'' = 0, \\ cn + gq\,.\,\beta\alpha'\alpha'' - fp\,.\,\alpha\beta'\beta'' = 0, \\ fp + gq + hr = 0. \end{array}$$

[* This Collection, vol. VII., p. 282.]

But substituting for a, b, c, l, m, n their values $\frac{1}{f}, \frac{1}{g}, \frac{1}{h}, \frac{1}{p}, \frac{1}{q}, \frac{1}{r}$, we have in all 8 equations for the determination of qr, rp, pq, gh, hf, fg; viz. if for greater convenience we introduce the new symbols $\mathfrak{A}, \mathfrak{B}, \mathfrak{C} = qr\alpha'\alpha'', rp\beta'\beta'', pq\gamma'\gamma''$, then the 8 equations are

$$\frac{1}{\beta\gamma} + \frac{\mathfrak{B}}{\beta} - \frac{\mathfrak{C}}{\gamma} = 0,$$

$$\frac{1}{\gamma\alpha} + \frac{\mathfrak{C}}{\gamma} - \frac{\mathfrak{A}}{\alpha} = 0,$$

$$\frac{1}{\alpha\beta} + \frac{\mathfrak{A}}{\alpha} - \frac{\mathfrak{B}}{\beta} = 0,$$

$$\frac{\alpha'\alpha''}{\mathfrak{A}} + \frac{\beta'\beta''}{\mathfrak{B}} + \frac{\gamma'\gamma''}{\mathfrak{C}} = 0,$$

$$\frac{1}{\beta\gamma} + hf.\frac{\mathfrak{B}}{\beta} - fg.\frac{\mathfrak{C}}{\gamma} = 0,$$

$$\frac{1}{\gamma\alpha} + fg.\frac{\mathfrak{C}}{\gamma} - gh.\frac{\mathfrak{A}}{\alpha} = 0,$$

$$\frac{1}{\alpha\beta} + gh.\frac{\mathfrak{A}}{\alpha} - hf.\frac{\mathfrak{B}}{\beta} = 0,$$

$$\frac{\alpha'\alpha''}{\mathfrak{A}gh} + \frac{\beta'\beta''}{\mathfrak{B}hf} + \frac{\gamma'\gamma''}{\mathfrak{C}fg} = 0.$$

But in virtue of the equation $\alpha + \beta + \gamma = 0$ the first four equations are equivalent to three equations only, and they determine $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}$, that is, p, q, r, which give at once l, m, n; and similarly the second four equations are equivalent to three equations only, and $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}$ being known they determine gh, hf, fg, that is, f, g, h, which give at once a, b, c: the identification of the two forms is thus completed.

Cambridge, 11th January, 1883.