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NOTE IN CONNEXION WITH THE HYPERELLIPTIC INTEGRALS OF THE FIRST ORDER.

[From *Crelle's Journal der Mathem.*, t. XCVIII. (1885), pp. 95, 96.]

In the early paper by Mr Weierstrass "Zur Theorie der *Abel*schen Functionen," *Crelle's Journal*, t. XLVII. (1854), pp. 289—306, we have pp. 302, 303, certain equations (43), and (stated to be deduced from them) an equation (49). Taking for greater simplicity $n=2$, the equations (43) written at full length are

$$(43)\ \begin{cases} K_{11}J_{12} - K_{12}J_{11} + K_{21}J_{22} - K_{22}J_{21} = 0, & K'_{11}J'_{12} - K'_{12}J'_{11} + K'_{21}J'_{22} - K'_{22}J'_{21} = 0, \\ K_{11}J'_{12} - K'_{12}J_{11} + K_{21}J'_{22} - K'_{22}J_{21} = 0, & K_{12}J'_{11} - K'_{11}J_{12} + K_{22}J'_{21} - K'_{21}J_{22} = 0, \\ K_{11}J'_{11} - K'_{11}J_{11} + K_{21}J'_{21} - K'_{21}J_{21} = \tfrac{1}{2}\pi, & K_{12}J'_{12} - K'_{12}J_{12} + K_{22}J'_{22} - K'_{22}J_{22} = \tfrac{1}{2}\pi; \end{cases}$$

viz. in the theory of the hyperelliptic functions depending on the radical

$$\sqrt{x-a_0\,.\,x-a_1\,.\,x-a_2\,.\,x-a_3\,.\,x-a_4},$$

these are relations between the eight integrals K of the first kind, and the eight integrals J of the second kind. Each equation contains both K's and J's, and there is not in the paper any express mention of a relation between the K's only, which occurs in Rosenhain's Memoir, and is a leading equation in the theory. But taking as before $n=2$, and for the G's which occur in (49) substituting their values as obtained from the preceding equations (46) and (47), the equation becomes

$$(49)\quad K_{11}K'_{21} - K_{21}K'_{11} + K_{12}K'_{22} - K_{22}K'_{12} = 0,$$

which is the equation in question: it is the equation $\omega_0 v_3 - \omega_3 v_0 + \omega_1 v_2 - \omega_2 v_1 = 0$ of Hermite's Memoir "Sur la théorie de la transformation des fonctions Abéliennes," *Comptes Rendus*, t. XL. (1855).

It is interesting to see how the equation (49) is derived from the equations (43). I write for greater convenience

$$\begin{matrix} K_{11}, & K_{12}, & K_{21}, & K_{22}, & K'_{11}, & K'_{12}, & K'_{21}, & K'_{22}, & J_{11}, & J_{12}, & J_{21}, & J_{22}, & J'_{11}, & J'_{12}, & J'_{21}, & J'_{22} \\ =A, & B, & C, & D, & A', & B', & C', & D', & \alpha, & \beta, & \gamma, & \delta, & \alpha', & \beta', & \gamma', & \delta'. \end{matrix}$$

The given equations then are

$$(43)\quad \begin{cases} A\beta - B\alpha + C\delta - D\gamma = 0, & A'\beta' - B'\alpha' + C'\delta' - D'\gamma' = 0, \\ A\beta' - B'\alpha + C\delta' - D'\gamma = 0, & A'\beta - B\alpha' + C'\delta - D\gamma' = 0, \\ A\alpha' - A'\alpha + C\gamma' - C'\gamma = \tfrac{1}{2}\pi, & B\beta' - B'\beta + D\delta' - D'\delta = \tfrac{1}{2}\pi; \end{cases}$$

and it is required to show that these lead to the relation

$$(49)\quad AC' - A'C + BD' - B'D = 0.$$

From the first and fourth equations, and from the second and third equations of (43), we deduce

$$(AC' - A'C)\,\beta + (C\alpha' - C'\alpha)\,B + (C\gamma' - C'\gamma)\,D = 0,$$
$$(AC' - A'C)\,\beta' + (C\alpha' - C'\alpha)\,B' + (C\gamma' - C'\gamma)\,D' = 0;$$

and again from the first and third equations, and from the second and fourth equations of (43), we deduce

$$(BD' - B'D)\,\alpha + (D\beta' - D'\beta)\,A + (D\delta' - D'\delta)\,C = 0,$$
$$(BD' - B'D)\,\alpha' + (D\beta' - D'\beta)\,A' + (D\delta' - D'\delta)\,C' = 0.$$

These pairs of equations give respectively

$$AC' - A'C : C\alpha' - C'\alpha : C\gamma' - C'\gamma = BD' - B'D : D\beta' - D'\beta : -(B\beta' - B'\beta),$$

and

$$AC' - A'C : C\alpha' - C'\alpha : -(A\alpha' - A'\alpha) = BD' - B'D : D\beta' - D'\beta : D\delta' - D'\delta;$$

whence putting for shortness $A\alpha' - A'\alpha,\ B\beta' - B'\beta,\ C\gamma' - C'\gamma,\ D\delta' - D'\delta = \mathrm{a, b, c, d}$, we have

$$\frac{AC' - A'C}{BD' - B'D} = -\frac{\mathrm{c}}{\mathrm{b}} = -\frac{\mathrm{a}}{\mathrm{d}};\ \text{whence } \mathrm{ab} = \mathrm{cd}.$$

But the last two of the equations (43) are

$$\mathrm{a} + \mathrm{c} = \tfrac{1}{2}\pi,\quad \mathrm{b} + \mathrm{d} = \tfrac{1}{2}\pi;$$

we have thus $\mathrm{a} + \mathrm{c} = \mathrm{b} + \mathrm{d},\ = \mathrm{b} + \frac{\mathrm{ab}}{\mathrm{c}},\ = \frac{\mathrm{b}}{\mathrm{c}}(\mathrm{a} + \mathrm{c})$; or, since $\mathrm{a} + \mathrm{c},\ = \tfrac{1}{2}\pi$, is not $= 0$, this gives $\mathrm{b} = \mathrm{c}$, whence also $\mathrm{a} = \mathrm{d}$, and we have

$$\frac{AC' - A'C}{BD' - B'D} = -1,$$

that is,

$$AC' - A'C + BD' - B'D = 0,$$

the required equation.

Cambridge, 10th September, 1884.