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ON A PROBLEM OF ANALYTICAL GEOMETRY.

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THE object of the present note is only to call attention to a problem of Analytical Geometry which presents itself in connexion with the reduction of an algebraical integral, and which is solved, pp. 21, 22 of Clebsch and Gordan's *Theorie der Abel'schen Functionen* (Leipzig, 1866); viz. the problem is, considering a line drawn through two given points of a curve $f=0$ of the order n, to find the equation of a curve $\Omega=0$ of the order $n-2$ passing through the remaining $n-2$ points of intersection of the line with the curve f, through the double points of f, and through as many other given points as are required for the determination of the curve. If, for instance, f is a quartic curve without double points, then Ω is the quadric curve which passes through the remaining two intersections of the line with Ω, and through three given points. Take (ξ_1, η_1, ζ_1), (ξ_2, η_2, ζ_2) the coordinates of the two given points on the curve f; (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3) for the coordinates of the three given points: and write $\Omega, =(a, b, c, f, g, h \between x, y, z)^2=0$ for the equation of the required curve. In the equation $f, =(x, y, z)^4=0$, write $x, y, z=\lambda\xi_1+\mu\xi_2,\ \lambda\eta_1+\mu\eta_2,\ \lambda\zeta_1+\mu\zeta_2$: we obtain an equation originally of the fourth order in (λ, μ), but which divides by $\lambda\mu$, and which when this factor is thrown out becomes

$$\alpha\lambda^2+\beta\lambda\mu+\gamma\mu^2=0\,;$$

where

$$\alpha=(\xi_1,\ \eta_1,\ \zeta_1)^3(\xi_2,\ \eta_2,\ \zeta_2),\quad =(\xi_2\partial_{\xi_1}+\eta_2\partial_{\eta_1}+\zeta_2\partial_{\zeta_1})f_1,$$

$$\beta=(\xi_1,\ \eta_1,\ \zeta_1)^2(\xi_2,\ \eta_2,\ \zeta_2)^2,$$

$$\gamma=(\xi_1,\ \eta_1,\ \zeta_1)(\xi_2,\ \eta_2,\ \zeta_2)^3,\quad =(\xi_1\partial_{\xi_2}+\eta_1\partial_{\eta_2}+\zeta_1\partial_{\zeta_2})f_2,$$

where for shortness f_1, f_2 are written to denote $f(\xi_1,\ \eta_1,\ \zeta_1)$, $f(\xi_2,\ \eta_2,\ \zeta_2)$ respectively.

The condition as to the two points obviously is that, making the same substitution $x, y, z = \lambda\xi_1 + \mu\xi_2, \lambda\eta_1 + \mu\eta_2, \lambda\zeta_1 + \mu\zeta_2$ in the equation $\Omega = (a, \ldots \between x, y, z)^2, = 0$, we must obtain the same quadric equation in (λ, μ). We have thus two conditions, which, introducing an indeterminate multiplier θ, are expressed by the three equations

$$\begin{aligned}(a, \ldots \between \xi_1, \eta_1, \zeta_1)^2 &= \theta\alpha,\\ (a, \ldots \between \xi_1, \eta_1, \zeta_1)(\xi_2, \eta_2, \zeta_2) &= \theta\beta,\\ (a, \ldots \between \xi_2, \eta_2, \zeta_2)^2 &= \theta\gamma.\end{aligned}$$

The conditions as to the three points are obviously

$$\begin{aligned}(a, \ldots \between x_1, y_1, z_1)^2 = 0,\\ (a, \ldots \between x_2, y_2, z_2)^2 = 0,\\ (a, \ldots \between x_3, y_3, z_3)^2 = 0,\end{aligned}$$

and these equations determine the ratios of a, b, c, f, g, h. But to complete the solution the convenient course is to regard the function $\Omega, = (a, \ldots \between x, y, z)^2$ as a quantity to be determined, and consequently to join to the foregoing the equation

$$(a, \ldots \between x, y, z)^2 = \Omega;$$

we have thus seven equations from which (a, b, c, f, g, h) may be eliminated, the result being expressed by means of a determinant of the seventh order

$$\left|\begin{array}{ll} (x, y, z)^2, & \Omega \\ (\xi_1, \eta_1, \zeta_1)^2, & \theta\alpha \\ (\xi_1, \eta_1, \zeta_1 \between \xi_2, \eta_2, \zeta_2), & \theta\beta \\ (\xi_2, \eta_2, \zeta_2)^2, & \theta\gamma \\ (x_1, y_1, z_1)^2, & 0 \\ (x_2, y_2, z_2)^2, & 0 \\ (x_3, y_3, z_3)^2, & 0 \end{array}\right| = 0,$$

viz. this is an equation of the form $A\Omega = \theta\nabla$, where A is a constant determinant of the sixth order (i.e. a determinant not involving x, y, z), ∇ a determinant of the seventh order, a quadric function of (x, y, z), obtained from the foregoing determinant by writing therein $\Omega = 0$ and $\theta = 1$: the multiplier θ is and remains arbitrary: but it is convenient to take it to be $= 1$, viz. we thus not only find the equation $\Omega = 0$, of the required conic, but we put a determinate value on the quadric function Ω itself. And this being so, it is to be remarked that, for $(x, y, z) = (\xi_1, \eta_1, \zeta_1)$, we have $\Omega = \alpha, = (\xi_2\partial_{\xi_1} + \eta_2\partial_{\eta_1} + \zeta_2\partial_{\zeta_1})f_1$: and so for $(x, y, z) = (\xi_2, \eta_2, \zeta_2)$, we have

$$\Omega = \gamma, = (\xi_1\partial_{\xi_2} + \eta_1\partial_{\eta_2} + \zeta_1\partial_{\zeta_2})f_2.$$