827.

ON THE NON-EUCLIDIAN PLANE GEOMETRY.

1. I CONSIDER the hyperbolic or Lobatschewskian geometry: this is a geometry such as that of the imaginary spherical surface $x^2 + y^2 + z^2 = -1$; and the imaginary surface may be *bent* (without extension or contraction) into the real surface considered by Beltrami, which I will call the Pseudosphere, viz. this is the surface of revolution defined by the equations $x = \log \cot \frac{1}{2}\theta - \cos\theta$, $\sqrt{y^2 + z^2} = \sin\theta$. We have on the imaginary spherical surface imaginary points corresponding to real points of the pseudosphere, and imaginary lines (arcs of great circle) corresponding to real lines (geodesics) of the pseudosphere, and, moreover, any two such imaginary points or lines of the imaginary spherical surface have a real distance or inclination equal to the corresponding distance or inclination on the pseudosphere. Thus the geometry of the pseudosphere, using the expression straight line to denote a geodesic of the surface, is the Lobatschewskian geometry; or rather I would say this in regard to the metrical geometry, or trigonometry, of the surface; for in regard to the descriptive geometry, the statement requires (as will presently appear) some qualification.

2. I would remark that this realisation of the Lobatschewskian geometry sustains the opinion that Euclid's twelfth axiom is undemonstrable. We may imagine rational beings living in a two-dimensional space and conceiving of space accordingly, that is, having no conception of a third dimension of space; this two-dimensional space need not however be a plane, and taking it to be the pseudospherical surface, the geometry to which their experience would lead them would be the geometry of this surface, that is, the Lobatschewskian geometry. With regard to our own two-dimensional space, the plane, I have, in my Presidential Address (B.A., Southport, 1883), [784], expressed the opinion that Euclid's twelfth axiom in Playfair's form of it does not need demonstration, but is part of our notion of space, of the physical space of our experience;

the space, that is, which we become acquainted with by experience, but which is the representation lying at the foundation of all physical experience.

3. I propose in the present paper to develope further the geometry of the pseudosphere. In regard to the name, and the subject generally, I refer to two memoirs by Beltrami, "Teoria fondamentale degli spazii di curvatura costante," *Annali di Matem.*, t. II. (1868—69), pp. 232—255, and "Saggio di interpretazione della geometria non-Euclidea," *Battaglini, Giorn. di Matem.*, t. VI. (1868), pp. 284—312, both translated, *Ann. de l'École Normale*, t. VI. (1869); in the last of these, he speaks of surfaces of constant negative curvature as "pseudospherical," and in a later paper, "Sulla superficie di rotazione che serve di tipo alle superficie pseudosferiche," *Battaglini, Giorn. di Matem.*, t. X. (1872), pp. 147—160, he treats of the particular surface which I have called the pseudosphere. The surface is mentioned, Note IV. of Liouville's edition of Monge's *Application de l'Analyse à la Géométrie* (1850), and the generating curve is there spoken of as "bien connue des géomètres."

4. In ordinary plane geometry, take (fig. 1) a line Bx, and on it a point B; from B, in any direction, draw the line BA; take upon it a point A, and from

Fig. 1.
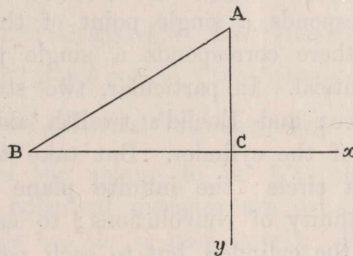
this point, at right angles to Bx, draw Ay, cutting it at C. We have thus a triangle ACB, right-angled at C; and we may denote the other angles, and the lengths of the sides, by A, B, c, a, b, respectively. In the construction of the figure, the length c and the angle B are arbitrary.

The plane is a surface which is homogeneous, isotropic, and palintropic, that is, whatever be the position of B, the direction of Bx, and the sense in which the angle B is measured, we have the same expressions for a, b as functions of c, B; these expressions, of course, are

$$a = c \cos B, \quad b = c \sin B.$$

But considering Ay as the initial line and AB, $= c$, as a line drawn from A at an inclination thereto $= A$, we have in like manner

$$b = c \cos A, \quad a = c \sin A,$$

and consequently $\cos A = \sin B$, $\sin A = \cos B$; whence $\sin (A + B) = 1$, $\cos (A + B) = 0$, and thence $A + B = $ a right angle, or $A + B + C = $ two right angles.

Hence also in any triangle ABC, drawing a perpendicular, say AD, from A to the side BC, and so dividing the triangle into two right-angled triangles, we prove that the sum $A + B + C$ of the angles is = two right angles, and we further establish the relations

$$a = b \cos C + c \cos B, \quad b = c \cos A + a \cos C, \quad c = a \cos B + b \cos A,$$

which are the fundamental formulæ of plane trigonometry; that is, we derive the metrical geometry or trigonometry of the plane from the two original equations $a = c \cos B$, $b = c \sin B$.

5. Supposing the plane bent in any manner, that is, converted into a developable surface or torse, and using the term straight line to denote a geodesic of the surface, then the straight line of the surface is in fact the form assumed, in consequence of the bending, by a straight line of the plane. The sides and angles of the rectilinear triangle ABC on the surface are equal to those of the rectilinear triangle ABC on the plane, and the metrical relations hold good without variation. But it is not *simpliciter* true that the descriptive properties of the torse are identical with those of the plane. This will be the case if the points of the plane and torse have with each other a (1, 1) correspondence, but not otherwise. For instance, consider a plane curve (such as the parabola or one branch of the hyperbola) extending from infinity to infinity, and let the torse be the cylinder having this curve for a plane section; then to each point of the plane there corresponds a single point of the cylinder; and conversely to each point of the cylinder there corresponds a single point of the plane; and the descriptive geometries are identical. In particular, two straight lines (geodesics) on the cylinder cannot inclose a space; and Euclid's twelfth axiom holds good in regard to the straight lines (geodesics) of the cylinder. But take the plane curve to be a closed curve, or (to fix the ideas) a circle; the infinite plane is bent into a cylinder considered as composed of an infinity of convolutions; to each point of the plane there corresponds a single point of the cylinder, but to each point of the cylinder an infinity of points of the plane; and the descriptive properties are in this case altered; the straight lines (geodesics) of the cylinder are helices; and we can through two given points of the cylinder draw, not only one, but an infinity of helices; any two of these will inclose a space. And even if instead of the geodesics we consider only the shortest lines, or helices of greatest inclination; yet even here for a pair of points on opposite generating lines of the cylinder, there are two helices of equal inclination, that is, two shortest lines inclosing a space. We have, in what precedes, an illustration in regard to the descriptive geometry of the pseudosphere; this is not identical with the Lobatschewskian geometry, but corresponds to it in a manner such as that in which the geometry of the surface of the circular cylinder corresponds to that of the plane.

6. The surface of the sphere is, like the plane, homogeneous, isotropic, and palintropic. We may on the spherical surface construct, as above, a right-angled triangle ABC, wherein the side c and the angle B are arbitrary; and (corresponding to the before-mentioned formulæ for the plane) we then have

$$\tan a = \tan c \cos B, \quad \sin b = \sin c \sin B,$$

whence also

$$\tan b = \tan c \cos A, \quad \sin a = \sin c \sin A.$$

We deduce from these

$$\frac{\tan^2 a}{\tan^2 c} + \frac{\sin^2 b}{\sin^2 c} = 1,$$

leading to $\cos^2 c = \cos^2 a \cos^2 b$; and then

$$\frac{\sin b}{\tan a} = \cos c \tan B, \quad \frac{\sin a}{\tan b} = \cos c \tan A,$$

giving

$$\cos a \cos b = \cos^2 c \tan A \tan B;$$

that is,

$$\tan A \tan B = \frac{1}{\cos a \cos b}, \text{ which is } > 1.$$

Hence $A + B >$ a right angle, or in the right-angled triangle ACB, the sum $A + B + C$ of the angles is $>$ two right angles. Whence also in any triangle ABC whatever, dividing it into two right-angled triangles by means of a perpendicular let fall from an angle on the opposite side, we have the sum $A + B + C$ of the angles $>$ two right angles. And we obtain, moreover,

$$a = \tan^{-1} (\tan c \cos B) + \tan^{-1} (\tan b \cos C),$$
$$b = \tan^{-1} (\tan a \cos C) + \tan^{-1} (\tan c \cos A),$$
$$c = \tan^{-1} (\tan b \cos A) + \tan^{-1} (\tan a \cos B),$$

which lead to all the formulæ of spherical trigonometry.

7. Suppose the radius of the sphere to be $1/\lambda$: then a, b, c being the lengths of the sides, the lengths in spherical measure are λa, λb, λc; and we must in the formulæ instead of a, b, c write λa, λb, λc respectively. In particular, for the imaginary sphere $x^2 + y^2 + z^2 = -1$, we have $\lambda = i$, and we must instead of a, b, c write ai, bi, ci respectively. The fundamental formulæ for the right-angled triangle thus become

$$\tanh a = \tanh c \cos B, \quad \sinh b = \sinh c \sin B,$$

and these lead to all the trigonometrical formulæ, viz. any one of these is deduced from the corresponding formula of spherical trigonometry by writing therein ai, bi, ci for a, b, c respectively; or, what is the same thing, by changing the circular functions of the sides into the corresponding hyperbolic functions.

In particular, for the right-angled triangle ACB, we have

$$\tan A \tan B = \frac{1}{\cosh a \cosh b},$$

which for a and b real is < 1, that is, $A + B <$ a right angle, or $A + B + C <$ two right angles, and thence also in any triangle whatever $A + B + C <$ two right angles. But the points A, B, C of any such triangle ABC on the imaginary sphere, and the lines BC, CA, AB which connect them, are imaginary: the meaning of the proof will better appear on passing to the pseudosphere.

8. We have to consider the imaginary spherical surface as *bent* into a real surface. This is, of course, an imaginary process, as any process must be which gives a transformation of imaginary points and lines into real points and lines; but the notion is not more difficult than that of the transformation of imaginary similarity, consisting in the substitution of ix, iy, iz for x, y, z respectively. We thus pass from imaginary points of the imaginary sphere $x^2 + y^2 + z^2 = -1$ to real points of the real sphere $x^2 + y^2 + z^2 = 1$; or, again, from imaginary points of either of the real hyperboloids $x^2 + y^2 - z^2 = -1$, $x^2 + y^2 - z^2 = 1$, to real points of the other of the same two real hyperboloids.

9. I consider the formulæ for the flexure of the imaginary sphere $X^2 + Y^2 + Z^2 = -1$, into the pseudosphere $x = \log \cot \frac{1}{2}\theta - \cos\theta$, $\sqrt{y^2 + z^2} = \sin\theta$: it would be allowable to dispense with Beltrami's subsidiary variables u, v, but I prefer to collect here all the formulæ. We have

$$X = \frac{-i}{\sqrt{1 - u^2 - v^2}}, \quad Y = \frac{u}{\sqrt{1 - u^2 - v^2}}, \quad Z = \frac{v}{\sqrt{1 - u^2 - v^2}},$$

values which give $X^2 + Y^2 + Z^2 = -1$. And observe that, taking u, v to be real magnitudes such that $u^2 + v^2 < 1$, we have X a pure imaginary, but Y and Z each of them real. We consider on the imaginary sphere points having such coordinates X, Y, Z; any such point corresponds as will immediately appear to a real point on the pseudosphere, and (the distances and angles being the same for the pseudosphere as for the original imaginary spherical surface) it hence appears that (notwithstanding that the points on the imaginary spherical surface, and the lines joining such points, are imaginary) the distances and angles on the imaginary spherical surface are real. Also

$$\sin\theta = \frac{1 - u}{\sqrt{1 - u^2 - v^2}}, \quad \phi = \frac{v}{1 - u},$$

and thence

$$iX - Y = \sin\theta, \quad iX + Y = \sin\theta\,(\phi^2 + \operatorname{cosec}^2\theta), \quad Z = \sin\theta \, . \, \phi.$$

Further

$$u = \frac{\phi^2 - 1 + \operatorname{cosec}^2\theta}{\phi^2 + 1 + \operatorname{cosec}^2\theta}, \quad v = \frac{2\phi}{\phi^2 + 1 + \operatorname{cosec}^2\theta},$$

$$x = \log \cot \tfrac{1}{2}\theta + \cos\theta, \quad y = \sin\theta \cos\phi, \quad z = \sin\theta \sin\phi.$$

10. We have $dX^2 + dY^2 + dZ^2$ and $dx^2 + dy^2 + dz^2$ each $= \cot^2\theta \, d\theta^2 + \sin^2\theta \, d\phi^2$. Writing P, $Q = iX - Y$, $iX + Y$ respectively, we in fact have

$$dX^2 + dY^2 + dZ^2 = -dP \, dQ + dZ^2,$$

where P, Q, $Z = \sin\theta$, $\phi^2 \sin\theta + \operatorname{cosec}\theta$, $\phi \sin\theta$ respectively; and thence

$$dZ = \sin\theta \, d\phi + \phi \cos\theta \, d\theta,$$
$$dP = \qquad\qquad \cos\theta \, d\theta,$$
$$dQ = 2\sin\theta \phi \, d\phi + (\phi^2 \cos\theta - \operatorname{cosec}\theta \cot\theta) \, d\theta,$$

giving the formula

$$dX^2 + dY^2 + dZ^2 = \cot^2 \theta \, d\theta^2 + \sin^2 \theta \, d\phi^2 \, ;$$

and then also

$$dx^2 + dy^2 + dz^2 = dx^2 + (d \sin \theta)^2 + \sin^2 \theta \, d\phi^2$$

$$= (\cos^2 \theta \cot^2 \theta + \cos^2 \theta) \, d\theta^2 + \sin^2 \theta \, d\phi^2 = \cot^2 \theta \, d\theta^2 + \sin^2 \theta \, d\phi^2.$$

Joining to these the differential expression in u, v, we have

$$dX^2 + dY^2 + dZ^2 = \frac{(1 - u^2 - v^2)(du^2 + dv^2) + (u \, du + v \, dv)^2}{(1 - u^2 - v^2)^2},$$

$$= \cot^2 \theta \, d\theta^2 + \sin^2 \theta \, d\phi^2,$$

$$= dx^2 + dy^2 + dz^2,$$

where the final equation $dX^2 + dY^2 + dZ^2 = dx^2 + dy^2 + dz^2$ shows that the imaginary sphere $X^2 + Y^2 + Z^2 = -1$ can be bent into the pseudosphere.

Observe that to given values of θ, ϕ there corresponds a single point on the pseudosphere, but not conversely; for if θ, ϕ be values corresponding to a given point, then corresponding to the same point we have θ, $\phi + n\pi$, where n is an arbitrary integer.

11. The geodesics of the imaginary spherical surface are, of course, its plane sections, any such section being determined by a linear equation $\alpha X + \beta Y + \gamma Z = 0$ between the coordinates X, Y, Z. Since for a point corresponding to a real point of the pseudosphere, X is a pure imaginary while Y and Z are real, we see that for a geodesic corresponding to a real geodesic of the pseudosphere we must have α a pure imaginary, β and γ real; and, in fact, writing as above, $P = iX - Y$, $Q = iX + Y$, and therefore conversely $X = \frac{1}{2}i(-P - Q)$, $Y = \frac{1}{2}(-P + Q)$, the equation $\alpha X + \beta Y + \gamma Z = 0$ becomes $(-\frac{1}{2}i\alpha - \frac{1}{2}\beta) P + (-\frac{1}{2}i\alpha + \frac{1}{2}\beta) Q + \gamma Z = 0$, which will then be of the form $AP + BQ + CZ = 0$, with real coefficients A, B, C: viz. we have

$$P, \ Q, \ Z = \sin \theta, \quad \sin \theta (\phi^2 + \operatorname{cosec}^2 \theta), \quad \sin \theta \cdot \phi \, ;$$

and the equation thus is

$$A + B (\phi^2 + \operatorname{cosec}^2 \theta) + C\phi = 0,$$

which is the equation for a geodesic (or straight line) on the pseudosphere. The equation $A + C\phi = 0$, that is, $\phi = $ const., is obviously that of a meridian.

12. If the geodesic pass through a given point θ_1, ϕ_1, we have, of course,

$$A + B (\phi_1^2 + \operatorname{cosec}^2 \theta_1) + C\phi_1 = 0,$$

and hence also the equation of a geodesic through the two points (θ_1, ϕ_1), (θ_2, ϕ_2) is

$$\begin{vmatrix} 1, & \phi^2 + \operatorname{cosec}^2 \theta, & \phi \\ 1, & \phi_1^2 + \operatorname{cosec}^2 \theta_1, & \phi_1 \\ 1, & \phi_2^2 + \operatorname{cosec}^2 \theta_2, & \phi_2 \end{vmatrix} = 0.$$

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We may for ϕ_1, ϕ_2 write $\phi_1 + 2n_1\pi$, $\phi_2 + 2n_2\pi$ respectively, n_1, n_2 being arbitrary integers; and it would thus at first sight appear that there could be drawn through the two points a doubly infinite series of geodesics. There is, in fact, a singly infinite system of geodesics: to show how this is, write for shortness Λ, Λ_1, Λ_2, α, α_1, α_2 for $\operatorname{cosec}^2\theta$, $\operatorname{cosec}^2\theta_1$, $\operatorname{cosec}^2\theta_2$, $2n\pi$, $2n_1\pi$, $2n_2\pi$ respectively; then the equation of the geodesic through the two points may be written

$$\begin{vmatrix} 1, & (\phi + \alpha)^2 + \Lambda, & \phi + \alpha \\ 1, & (\phi_1 + \alpha_1)^2 + \Lambda_1, & \phi_1 + \alpha_1 \\ 1, & (\phi_2 + \alpha_2)^2 + \Lambda_2, & \phi_2 + \alpha_2 \end{vmatrix} = 0,$$

where the constant $\alpha = 2n\pi$ may be disposed of so as to simplify the formula as much as may be: it is what I have called an apoclastic constant. Taking β an arbitrary value, this may be transformed into

$$\begin{vmatrix} 1, & (\phi + \alpha + \beta)^2 + \Lambda, & \phi + \alpha + \beta \\ 1, & (\phi_1 + \alpha_1 + \beta)^2 + \Lambda_1, & \phi_1 + \alpha_1 + \beta \\ 1, & (\phi_2 + \alpha_2 + \beta)^2 + \Lambda_2, & \phi_2 + \alpha_2 + \beta \end{vmatrix} = 0,$$

and then assuming $\alpha = \alpha_1$, $\beta = -\alpha_1$, this becomes

$$\begin{vmatrix} 1, & \phi^2 + \Lambda, & \phi \\ 1, & \phi_1^2 + \Lambda_1, & \phi_1 \\ 1, & (\phi_2 + \alpha_2 - \alpha_1)^2 + \Lambda_2, & \phi_2 + \alpha_2 - \alpha_1 \end{vmatrix} = 0,$$

which is what the equation

$$\begin{vmatrix} 1, & \phi^2 + \Lambda, & \phi \\ 1, & \phi_1^2 + \Lambda_1, & \phi_1 \\ 1, & \phi_2^2 + \Lambda_2, & \phi_2 \end{vmatrix} = 0,$$

becomes on changing only ϕ_2 into $\phi_2 + \alpha_2 - \alpha_1$, that is, $\phi_2 + 2k_2\pi$, where k_2 is an arbitrary integer. We have thus through the two points a singly infinite series of geodesic lines; in general, only one of these is a shortest line, but for points on opposite meridians there are two equal shortest lines.

13. For the distance between two points (θ_1, ϕ_1) and (θ_2, ϕ_2) on the pseudosphere, taking (X_1, Y_1, Z_1) and (X_2, Y_2, Z_2) for the corresponding points on the imaginary sphere, and writing as above P_1, $Q_1 = iX_1 - Y_1$, $iX_1 + Y_1$; P_2, $Q_2 = iX_2 - Y_2$, $iX_2 + Y_2$, we have

$$\cosh\delta = - X_1X_2 - Y_1Y_2 - Z_1Z_2,$$

$$= \tfrac{1}{2}(P_1Q_2 + P_2Q_1) - Z_1Z_2,$$

$$= \sin\theta_1\sin\theta_2\{\tfrac{1}{2}(\phi_1^2 + \operatorname{cosec}^2\theta_1) + \tfrac{1}{2}(\phi_2^2 + \operatorname{cosec}^2\theta_2) - \phi_1\phi_2\},$$

$$= \tfrac{1}{2}\sin\theta_1\sin\theta_2(\phi_2 - \phi_1)^2 + 1 + \frac{\tfrac{1}{2}(\sin\theta_2 - \sin\theta_1)^2}{\sin\theta_1\sin\theta_2}.$$

Observe here that, writing θ_2, $\phi_2 = \theta_1 + d\theta_1$, $\phi_1 + d\phi_1$, and therefore δ small so that $\cosh \delta = 1 + \frac{1}{2}\delta^2$, we obtain

$$\delta^2 = \sin^2 \theta_1 d\phi_1{}^2 + \cot^2 \theta_1 d\theta_1{}^2,$$

agreeing with the expression for $dx^2 + dy^2 + dz^2$. If in the form first obtained we write $\Lambda_1 = \operatorname{cosec}^2 \theta_1$, $\Lambda_2 = \operatorname{cosec}^2 \theta_2$, we find

$$\cosh \delta = \frac{\frac{1}{2}}{\sqrt{\Lambda_1 \Lambda_2}} \{\phi_1{}^2 + \Lambda_1 + \phi_2{}^2 + \Lambda_2 - 2\phi_1\phi_2\},$$

which is a convenient form.

In like manner, to find the mutual inclination of the two geodesics

$$A_1 + B_1 (\phi^2 + \operatorname{cosec}^2 \theta) + C_1\phi = 0,$$

$$A_2 + B_2 (\phi^2 + \operatorname{cosec}^2 \theta) + C_2\phi = 0,$$

these correspond to the plane sections

$$A_1 P + B_1 Q + C_1 Z = 0, \quad A_2 P + B_2 Q + C_2 Z = 0,$$

that is,

$$(A_1 + B_1) iX + (- A_1 + B_1) Y + C_1 Z = 0, \quad (A_2 + B_2) iX + (- A_2 + B_2) Y + C_2 Z = 0,$$

of the imaginary sphere: and we thence find

$$\cos \Omega = \frac{C_1 C_2 - 2 (A_1 B_2 + A_2 B_1)}{\sqrt{C_1{}^2 - 4 A_1 B_1} \sqrt{C_2{}^2 - 4 A_2 B_2}}.$$

14. Suppose that the two geodesics meet in the point θ_0, ϕ_0: then writing for shortness $\operatorname{cosec}^2 \theta = \Lambda$, and therefore $\operatorname{cosec}^2 \theta_0 = \Lambda_0$, we have

$$A_1 + B_1 (\phi_0{}^2 + \Lambda_0) + C_1\phi_0 = 0,$$

$$A_2 + B_2 (\phi_0{}^2 + \Lambda_0) + C_2\phi_0 = 0.$$

Suppose that the meridian through this point is

$$A_3 + B_3 (\phi^2 + \Lambda) + C_3\phi = 0;$$

then $B_3 = 0$, $A_3 + C_3\phi_0 = 0$. Take Ω_1, Ω_2, for the inclinations to this meridian of the two geodesics respectively; then

$$\cos \Omega_1 = \frac{C_1 C_3 - 2 A_3 B_1}{\sqrt{C_1{}^2 - 4 A_1 B_1} \cdot C_3} = \frac{C_1 + 2 B_1 \phi_0}{\sqrt{C_1{}^2 - 4 A_1 B_1}},$$

whence

$$\sin \Omega_1 = \frac{2 B_1 \sqrt{\Lambda_0}}{\sqrt{C_1{}^2 - 4 A_1 B_1}},$$

and similarly

$$\cos \Omega_2 = \frac{C_2 + 2 B_2 \phi_0}{\sqrt{C_2{}^2 - 4 A_2 B_2}},$$

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whence

$$\sin \Omega_2 = \frac{2B_2 \sqrt{\Lambda_0}}{\sqrt{C_2^2 - 4A_2 B_2}}.$$

We thence obtain

$$\cos (\Omega_1 - \Omega_2) = \frac{C_1 C_2 + 2\phi_0 (B_1 C_2 + B_2 C_1) + 4B_1 B_2 (\phi_0^2 + \Lambda_0)}{\sqrt{C_1^2 - 4A_1 B_1} \sqrt{C_2^2 - 4A_2 B_2}},$$

which is

$$= \frac{C_1 C_2 - 2 (A_1 B_2 + A_2 B_1)}{\sqrt{C_1^2 - 4A_1 B_1} \sqrt{C_2^2 - 4A_2 B_2}} = \cos \Omega,$$

as above, the equality of the two numerators depending on the identity

$$\{A_1 + B_1 (\phi_0^2 + \Lambda_0) + C_1 \phi_0\} B_2 + \{A_2 + B_2 (\phi_0^2 + \Lambda_0) + C_2 \phi_0\} B_1 = 0.$$

In particular, if we consider the two geodesics

$$\phi^2 + \operatorname{cosec}^2 \theta - \operatorname{cosec}^2 \theta_1 + C_1 \phi = 0, \quad \phi = 0,$$

the second of which may be considered as representing any meridian section of the pseudosphere, and the first is an arbitrary geodesic meeting this at the point $\theta = \theta_1$, $\phi = 0$, then the formula for the inclination is

$$\cos \Omega = \frac{C_1}{\sqrt{C_1^2 + 4 \operatorname{cosec}^2 \theta_1}}.$$

Hence also, $\cos \Omega = 0$, or $\Omega = 90°$, if $C_1 = 0$: viz. we have $\phi^2 + \operatorname{cosec}^2 \theta - \operatorname{cosec}^2 \theta_1 = 0$ for the equation of the geodesic through the point $\theta = \theta_1$, $\phi = 0$, at right angles to the meridian section $\phi = 0$.

15. Consider a right-angled triangle ACB, where the points A, C are on the meridian $\phi = 0$, and write $(\theta_1, 0; \Lambda_1 = \operatorname{cosec}^2 \theta_1)$, $(\theta_2, \phi_2; \Lambda_2 = \operatorname{cosec}^2 \theta_2)$, $(\theta_3, 0; \Lambda_3 = \operatorname{cosec}^2 \theta_3)$, for the points A, B, C respectively. Then if the equations are—

for the side BC, $A_1 + B_1 (\phi^2 + \Lambda) + C_1 \phi = 0$, we have $C_1 = 0$,

$A_1 + B_1 (\phi_2^2 + \Lambda_2) = 0$, $A_1 + B_1 \Lambda_3 = 0$, whence $\phi_2^2 + \Lambda_2 = \Lambda_3$;

for the side CA, $A_2 + B_2 (\phi^2 + \Lambda) + C_2 \phi = 0$, we have $A_2 = 0$, $B_2 = 0$;

for the side AB, $A_3 + B_3 (\phi^2 + \Lambda) + C_3 \phi = 0$,

$$\text{we have } A_3 + B_3 (\phi_2^2 + \Lambda_2) + C_3 \phi_2 = 0, \quad A_3 + B_3 \Lambda_1 = 0.$$

Observing that $\phi_1 = \phi_3 = 0$, we have

$$\cosh a = \frac{1}{2 \sqrt{\Lambda_2 \Lambda_3}} (\phi_2^2 + \Lambda_2 + \Lambda_3),$$

$$\cosh b = \frac{1}{2 \sqrt{\Lambda_1 \Lambda_3}} (\Lambda_3 + \Lambda_1),$$

$$\cosh c = \frac{1}{2 \sqrt{\Lambda_1 \Lambda_2}} (\phi_2^2 + \Lambda_1 + \Lambda_2);$$

or, reducing these by the relation $\phi_2^2 + \Lambda_2 = \Lambda_3$, they become

$$\cosh a = \frac{\sqrt{\Lambda_3}}{\sqrt{\Lambda_2}}, \quad \text{whence} \ \sinh a = \frac{\sqrt{\Lambda_3 - \Lambda_2}}{\sqrt{\Lambda_2}}, \quad \tanh a = \frac{\sqrt{\Lambda_3 - \Lambda_2}}{\sqrt{\Lambda_3}};$$

$$\cosh b = \frac{\Lambda_1 + \Lambda_3}{2\sqrt{\Lambda_1 \Lambda_3}}, \quad \text{,,} \ \sinh b = \frac{\Lambda_1 - \Lambda_3}{2\sqrt{\Lambda_1 \Lambda_3}}, \quad \tanh b = \frac{\Lambda_1 - \Lambda_3}{\Lambda_1 + \Lambda_3};$$

$$\cosh c = \frac{\Lambda_1 + \Lambda_3}{2\sqrt{\Lambda_1 \Lambda_2}}, \quad \text{,,} \ \sinh c = \frac{\sqrt{(\Lambda_1 + \Lambda_3)^2 - 4\Lambda_1 \Lambda_2}}{2\sqrt{\Lambda_1 \Lambda_2}}, \quad \tanh c = \frac{\sqrt{(\Lambda_1 + \Lambda_3)^2 - 4\Lambda_1 \Lambda_2}}{\Lambda_1 + \Lambda_3}.$$

We have, moreover,

$$\cos B = \frac{C_1 C_3 - 2(A_1 B_3 + A_3 B_1)}{\sqrt{C_1^2 - 4A_1 B_1} \sqrt{C_3^2 - 4A_3 B_3}} = \frac{-2(A_1 B_3 + A_3 B_1)}{\sqrt{-4A_1 B_1} \sqrt{C_3^2 - 4A_3 B_3}},$$

which, writing $A_3 = -B_3 \Lambda_1$ and $A_1 = -B_1 \Lambda_3$, becomes

$$\cos B = \frac{B_3(\Lambda_1 + \Lambda_3)}{\sqrt{\Lambda_3} \sqrt{C_3^2 - 4A_3 B_3}};$$

or, further reducing by means of

$$\begin{aligned}
\phi_2^2 (C_3^2 - 4A_3 B_3) &= B_3^2 (\phi_2^2 + \Lambda_2 - \Lambda_1)^2 + 4\phi_2^2 B_3^2 \Lambda_1^2 \\
&= B_3^2 \{ (\phi_2^2 + \Lambda_2 - \Lambda_1)^2 + 4\phi_2^2 \Lambda_1 \} \\
&= B_3^2 \{ (\Lambda_3 + \Lambda_1)^2 + 4\Lambda_1(\Lambda_3 - \Lambda_2) \} \\
&= B_3^2 \{ (\Lambda_3 + \Lambda_1)^2 - 4\Lambda_1 \Lambda_2 \},
\end{aligned}$$

this becomes

$$\cos B = \frac{(\Lambda_1 + \Lambda_3)\sqrt{\Lambda_3 - \Lambda_2}}{\sqrt{\Lambda_3} \sqrt{(\Lambda_1 + \Lambda_3)^2 - 4\Lambda_1 \Lambda_2}},$$

whence

$$\sin B = \frac{\sqrt{\Lambda_2}(\Lambda_1 - \Lambda_3)}{\sqrt{\Lambda_3} \sqrt{(\Lambda_1 + \Lambda_3)^2 - 4\Lambda_1 \Lambda_2}};$$

and with these values we verify

$$\tanh a = \tanh c . \cos B,$$
$$\sinh b = \sinh c . \sin B,$$

which are the expressions for the sides BC, CA, in terms of the length BA, $= c$ and angle B, which are arbitrary. I have not thought it necessary to give the direct verification of these equations for a more general position of the right-angled triangle: we already know, and it appears *à posteriori* by the following number, that the verification really extends to any right-angled triangle whatever on the surface.

16. The pseudosphere is homogeneous, isotropic, and palintropic, viz. this is the case when bending is allowed; in other words, the surface is applicable upon itself, with three degrees of freedom. Considering any infinitesimal linear element Ax, the point A may be brought to coincide with an arbitrary point A' of the surface, and

the element Ax to lie in an arbitrary direction $A'x'$ through A'; the area about A will then coincide with the area about A'. The analytical theory is at once derived from that for the sphere, viz. we have a rectangular transformation

	X	Y	Z
X_1	α	β	γ
Y_1	α'	β'	γ'
Z_1	α''	β''	γ''

where the coefficients are such that identically

$$X_1{}^2 + Y_1{}^2 + Z_1{}^2 = X^2 + Y^2 + Z^2;$$

in fact, the coefficients are connected by six equations only, the system thus depending on three arbitrary parameters. If, as before, we write P_1, Q_1, P, Q, for $iX_1 - Y_1$, $iX_1 + Y_1$, $iX - Y$, $iX + Y$ respectively, then the relation is readily found to be

	P	Q	Z
P_1	$\frac{1}{2}(\alpha + i\alpha' - i\beta + \beta')$	$\frac{1}{2}(\alpha + i\alpha' + i\beta - \beta')$	$i\gamma - \gamma'$
Q_1	$\frac{1}{2}(\alpha - i\alpha' - i\beta - \beta')$	$\frac{1}{2}(\alpha - i\alpha' + i\beta + \beta')$	$i\gamma + \gamma'$
Z_1	$\frac{1}{2}(-i\alpha'' - \beta'')$	$\frac{1}{2}(-i\alpha'' + \beta'')$	γ''

this being read according to the lines only $P_1 = \frac{1}{2}(\alpha + i\alpha' - i\beta + \beta')P + \&c.$, not according to the columns: in order that the coefficients may be real, we must have α, β', γ', β'', γ'', real, β, γ, α', α'' pure imaginaries.

Writing the equations in the form

	P	Q	Z
P_1	A	B	C
Q_1	A'	B'	C'
Z_1	A''	B''	C''

viz.

$$P_1 = AP + BQ + CZ, \&c.,$$

it would be possible to deduce the equations which connect the new coefficients; but these are more easily obtained from the consideration that we must have identically $P_1Q_1 - Z_1^2 = PQ - Z^2$; the equations are thus found to be

$$A''^2 - AA' = 0, \qquad B''^2 - BB' = 0, \qquad C''^2 - CC' = 1,$$
$$2A''B'' - AB' - A'B = -1, \qquad 2A''C'' - AC' - A'C = 0, \qquad 2B''C'' - BC' - B'C = 0.$$

17. The general theory of the transformation of a quadric function into itself enables us to express the coefficients in terms of three arbitrary parameters. There is no difficulty in working out the formulæ, and we finally obtain

$$\Omega P_1 = - \quad (\nu + 1)^2\, P - \qquad \lambda^2\, Q + \qquad 2\lambda\,(\nu + 1)\, Z,$$
$$\Omega Q_1 = - \qquad \mu^2\, P - \quad (\nu - 1)^2\, Q + \qquad 2\mu\,(\nu - 1)\, Z,$$
$$\Omega Z_1 = - \mu\,(\nu + 1)\, P - \lambda\,(\nu - 1)\, Q + (-1 + \nu^2 + \lambda\mu)\, Z;$$

and conversely

$$\Omega P = - \quad (\nu - 1)^2\, P_1 - \qquad \lambda^2\, Q_1 + \qquad 2\lambda\,(\nu - 1)\, Z_1,$$
$$\Omega Q = - \qquad \mu^2\, P_1 - \quad (\nu + 1)^2\, Q_1 + \qquad 2\mu\,(\nu + 1)\, Z_1,$$
$$\Omega Z = - \mu\,(\nu - 1)\, P_1 - \lambda\,(\nu + 1)\, Q_1 + (1 + \nu^2 + \lambda\mu)\, Z_1,$$

where $\Omega = -1 + \nu^2 - \lambda\mu$: it can be at once verified that each of the two sets of formulæ does, in fact, give $P_1Q_1 - Z_1^2 = PQ - Z^2$.

18. The pseudosphere is a surface of revolution having for its meridian section the curve $x = \log \cot \tfrac{1}{2}\theta - \cos\theta$, $y = \sin\theta$. This is a curve symmetrical in regard to the axis of y; and we obtain the portion of it lying on the positive side of this axis, by giving to θ the series of values $\theta = 0$ to $\theta = 90°$; for $\theta = 0$, we have $y = 0$, $x = \infty$, or the axis of x is an asymptote; for $\theta = 90°$, $x = 0$, $y = 1$, the point being a cusp of the curve. The geometrical definition is that the portion of the tangent included between the curve and the axis of x has the constant length $= 1$; the inclination of

Fig. 2.
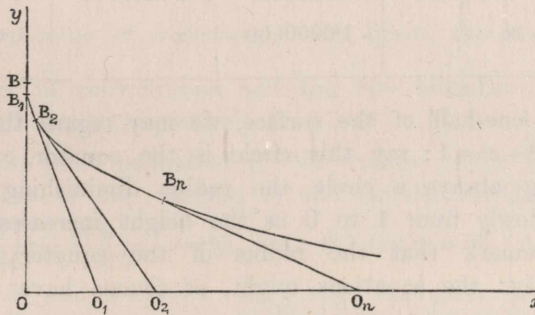
the tangent is in fact $= \theta$. We have $dx = \dfrac{\cos^2\theta\, d\theta}{\sin\theta}$, $dy = \cos\theta\, d\theta$; and thence $ds = \cot\theta\, d\theta$, and the length in question is $\dfrac{y\, ds}{dy} = 1$. The curve may be constructed graphically: take (fig. 2) the distance $BO = 1$, on OB, B_1 very near to B, and then $B_1O_1 = 1$; on

O_1B_1, B_2 very near to B_1, and then $B_2O_2 = 1$, and so on; the curve is shown on a larger scale in fig. 3, p. 235.

But the curve may also be laid down numerically; writing $\alpha = \frac{1}{2}\pi - \theta$, so that α is the inclination of the tangent to the axis of y, we have $x = \log \tan(\frac{1}{4}\pi + \frac{1}{2}\alpha) - \sin\alpha$, $y = \cos\alpha$, where $\log\tan(\frac{1}{4}\pi + \frac{1}{2}\alpha)$, the hyperbolic logarithm (which has been the signification of log throughout), is the function tabulated Tab. IV., Legendre's *Traité des Fonctions Elliptiques*, t. II. pp. 256—259.

We may hence obtain the values of the coordinates as follows:—

$\alpha = 90° - \theta$	$\log\tan(\frac{1}{4}\pi + \frac{1}{2}\alpha)$	$-\sin\alpha$	x	$y = \cos\alpha$
0	0·0000000	$-$ 0·0000000	0·0000000	1·0000000
10°	0·1754258	0·1736482	0·0017776	0·9848078
20°	0·3563785	0·3420201	0·0143584	0·9396926
30°	0·5493061	0·5000000	0·0493061	0·8660254
40°	0·7629096	0·6427876	0·1201220	0·7660444
50°	1·0106831	0·7660444	0·2446387	0·6427876
60°	1·3169578	0·8660254	0·4509324	0·5000000
70°	1·7354151	0·9396926	0·7957225	0·3563785
80°	2·4362460	0·9848078	1·4514382	0·1754258
85°	3·1313013	0·9961947	2·1351066	0·0871557
86°	3·3546735	0·9975641	2·3571019	0·0697565
87°	3·6425333	0·9986295	2·6439038	0·0523360
88°	4·0481254	0·9993908	3·0487346	0·0348995
89°	4·7413487	0·9998477	3·7415010	0·0174524
90°	∞	$-$ 1·0000000	∞	0·0000000

Attending only to one-half of the surface, we may regard the surface as standing on the circular base $y^2 + z^2 = 1$: say this circle is the equator, or the unit-circle: the horizontal section being always a circle, the radius diminishing at first rapidly and then more and more slowly from 1 to 0 as the height increases from 0 to ∞. It is hardly necessary to remark that the radius of the equator is any given length whatever, taken as unity: the equations might, of course, have been written

$$x = c \{\log \cot \tfrac{1}{2}\theta - \cos\theta\}, \quad \sqrt{y^2 + z^2} = c\sin\theta,$$

but there would have been no gain of generality in this.

19. The geodesics are as already seen given by an equation

$$A + B(\phi^2 + \operatorname{cosec}^2\theta) + C\phi = 0.$$

If $B = 0$, we have $A + C\phi = 0$, that is, $\phi =$ const., which belongs to the meridians; if B be not $= 0$, we may by a mere change of ϕ, that is, of the initial meridian, reduce the form to $A + B(\phi^2 + \operatorname{cosec}^2 \theta) = 0$, which is the equation of a geodesic cutting at right angles the meridian $\phi = 0$; writing herein $\sin \theta = \dfrac{1}{r}$, we have $A + B\left(\phi^2 + \dfrac{1}{r^2}\right) = 0$, which is the equation in the polar coordinates r, ϕ of the projection of the geodesic on the equatorial plane $x = 0$: putting herein for greater convenience $B = -Ak^2$, we have $r^2 = \dfrac{k^2}{1 - k^2\phi^2}$: we require only such portions of the curves as lie within the unit-circle, and need therefore attend only to those for which k is not greater than 1, and in any such curve consider ϕ as extending from $\phi = 0$ to $\phi = \pm \dfrac{\sqrt{1 - k^2}}{k}$: writing this last value $= \pm \gamma$, we have $k = \dfrac{1}{\sqrt{1 + \gamma^2}}$; if $\gamma < \pi$, that is, $k < \dfrac{1}{\sqrt{1 + \pi^2}}$, the curve is a mere arc cutting at right angles (at the distance $r = k$ from the centre) the meridian $\phi = 0$, and extending itself out on each side to meet the unit-circle in the points $\phi = \gamma$, $\phi = -\gamma$ respectively; in the case $\gamma = \pi$, that is, $k = \dfrac{1}{\sqrt{1 + \pi^2}}$, the two points $\phi = \pm \gamma$ come together at the point $\phi = \pi$, or the curve becomes a loop; and for larger values, $k = \dfrac{1}{\sqrt{1 + \pi^2}}$ to $\dfrac{1}{\sqrt{1 + 4\pi^2}}$, we have the two branches crossing each other on the meridian $\phi = \pi$ at the distance $r = \dfrac{k}{\sqrt{1 - k^2\pi^2}}$ from the centre and then extending themselves in the opposite semicircles, so as to meet the unit-circle at the points $\phi = \pm \gamma$. And we have thus another critical value $k = \dfrac{1}{\sqrt{1 + 4\pi^2}}$, for which the two branches having thus crossed each other come to unite themselves at the point $\phi (= 2\pi) = 0$ of the unit-circle; and in like manner the critical values $\dfrac{1}{\sqrt{1 + 9\pi^2}}$, $\dfrac{1}{\sqrt{1 + 16\pi^2}}$, &c.: for a value of k between such limits, the branch is a spiral having a determinate number of convolutions, and the two branches cross each other always on the radii $\phi = 0$ and $\phi = \pi$ respectively.

20. Let ψ denote the inclination of the radius vector to the normal, or, what is the same thing, that of the element of the circular arc to the tangent; we have $\tan \psi = \dfrac{dr}{r\,d\phi}$, and $\dfrac{dr}{r\,d\phi} = \dfrac{k^2\phi}{1 - k^2\phi^2}$, $= r^2\phi$, that is, $\tan \psi = r^2$. At the intersection with the unit section $r = 1$, and therefore $\tan \psi = \phi$; moreover putting $k = \cos \kappa$, so that the equation of the curve now is $r^2 = \dfrac{\cos^2 \kappa}{1 - \phi^2 \cos^2 \kappa}$, then for $r = 1$ we have $\phi = \tan \kappa$; and hence at the intersection with the unit-circle $\psi = \kappa$, that is, as k decreases from $k = 1$, or k increases from $k = 0$, the angle at which each curve cuts the unit-circle is always $= \kappa$, and thus this angle continually increases from $\kappa = 0$; for $k = \dfrac{1}{\sqrt{1 + \pi^2}} = \cos \kappa$,

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and therefore $\tan \kappa = \pi$, we have $\kappa = 72° 20'$ nearly: the complement hereof $17° 40'$ is thus the angle at which each branch of the loop cuts the meridian $\phi = \pi$.

21. To obtain another datum convenient in tracing the curve, I write $\phi = \phi_0 = \tan \kappa$ for the value of ϕ at the unit-circle; and introducing for the moment the rectangular coordinates $X = r \sin \phi$, $Y = 1 - r \cos \phi$, then we easily find

$$\frac{dY}{dX} = \frac{r \sin \phi - r^3 \phi \cos \phi}{r \cos \phi + r^3 \phi \sin \phi};$$

and thence, for the equation of the tangent at the point on the unit-circle,

$$(y - 1 + \cos \phi_0) = \frac{\sin \phi_0 - \phi_0 \cos \phi_0}{\cos \phi_0 + \phi_0 \sin \phi_0} (x - \sin \phi_0).$$

For the tangent at the point of intersection with the radius $\phi = 0$, or say the apse, we have $y = 1 - \cos \kappa$; and hence, at the intersection of the two tangents,

$$x = \sin \phi_0 + \frac{\cos \phi_0 + \phi_0 \sin \phi_0}{\sin \phi_0 - \phi_0 \cos \phi_0} (\cos \phi_0 - \cos \kappa)$$

$$= \frac{1 - \cos \kappa (\cos \phi_0 + \phi_0 \sin \phi_0)}{\sin \phi_0 - \phi_0 \cos \phi_0},$$

which, putting therein $\phi_0 = \tan \kappa$, becomes

$$= \frac{\cos \kappa \{1 - \cos (\phi_0 - \kappa)\}}{\sin (\phi_0 - \kappa)} = \cos \kappa \tan \tfrac{1}{2} (\phi_0 - \kappa),$$

where ϕ_0 is given in terms of κ by the just-mentioned equation $\phi_0 = \tan \kappa$. We have $y = 1 - \cos \kappa$, $x = \cos \kappa \tan \tfrac{1}{2} (\phi_0 - \kappa)$, for the locus of the intersection of the two tangents; this is easily seen to be a curve having a cusp at the unit-circle.

22. Fig. 3 shows the curves for the values

$\phi_0 =$	$\tan \kappa$	$\kappa =$
$30° = \tfrac{1}{6}\pi$	0·5235988	27° 38'
60	1·0471976	46 19
90	1·5707963	57 31
120	2·0943941	64 29
150	2·6179939	69 5
$180 = \pi$	3·1415926	72 20

We construct and graduate the unit-circle; draw to it a tangent at 0°, and measuring off from 0 a distance equal to the semi-circumference, graduate this in like manner in equal parts 0° to 180°; then to find the curve belonging, for instance, to

$\phi_0 = 90°$, we join with the centre of the circle the point $90°$ of the tangent, thus determining on the unit-circle a point belonging to the angle $\kappa = 57° 31'$; at this point we draw parallel to the tangent a line which is the tangent at the lowest

Fig. 3.
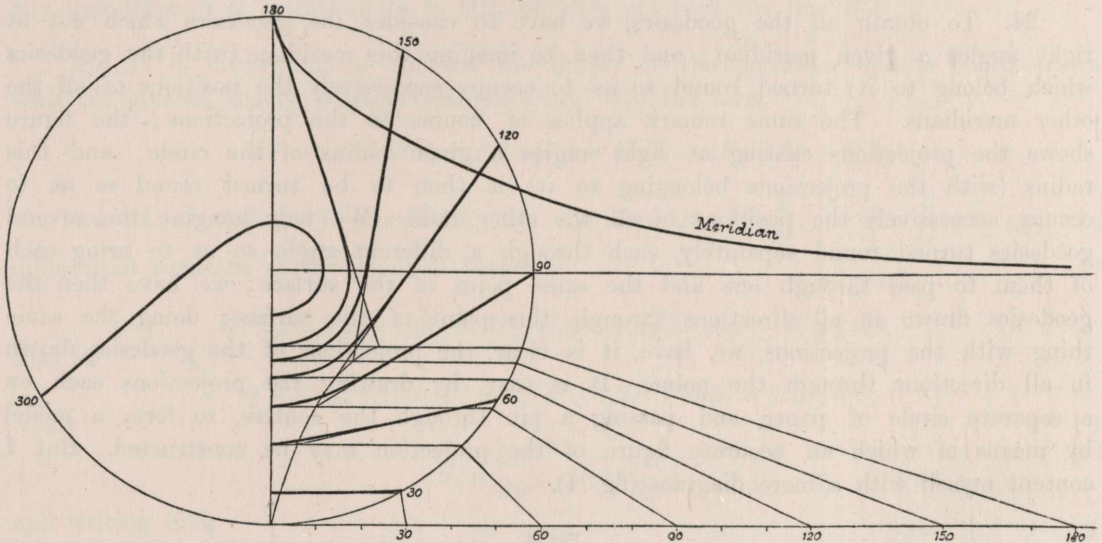
point; the curve passes through the point $90°$ on the unit-circle, and there cuts the circle at the angle $\kappa = 57° 31'$ (or, what is the same thing, the radius at the complementary angle), and we have thus the tangent at the point $90°$ of the unit-circle; it will be noticed that this meets the tangent at the apse at a point near to this apse, so that the arc as determined by the two tangents is for a large part of its course nearly a right line; this is still more the case for smaller values of ϕ_0 or κ, while for larger values the deviation increases, but in the neighbourhood of the unit-circle the form is always nearly rectilinear.

I show in the same figure the form of the curve for $\phi_0 = 300°$, $= 5\cdot2359877$, $= \tan \kappa$, that is, $\kappa = 79° 11'$, $r = \cos \kappa = 0\cdot1876670$, the value at the apse: the construction for the tangent at the unit-circle is the same as before, but in order to lay down the curve with tolerable accuracy we require also the value of r at the node on the meridian $\phi = 180°$; this is, of course, given by $r = \dfrac{\cos \kappa}{\sqrt{1 - \pi^2 \cos^2 \kappa}}$, that is, $r = \dfrac{\cos \kappa}{\cos \alpha}$, if $\pi \cos \kappa = \sin \alpha$; whence without difficulty $r = 0\cdot23236$, the value at the node.

23. The curves shown in the figure are projections upon the plane of the unit-circle, viz. they are the projections on this plane of the geodesics, which cut at right angles a given meridian; but bearing in mind the form of the meridian, it is easy, by means of the projection, to understand the actual forms on the surface of the pseudosphere. A point near the centre of the figure represents a point high up on the surface; and in any radius the portions near the centre are the more foreshortened in the figure, and represent greater distances on the surface. Each geodesic

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cutting the meridian at right angles at the apse descends symmetrically on the two sides, reaches ultimately—it may be after many convolutions—the unit-circle; the meridian itself is a limiting or special form of geodesic. The unit-circle is not properly a geodesic, but it is an envelope of geodesics.

24. To obtain all the geodesics, we have to consider the geodesics which cut at right angles a given meridian; and then to imagine this meridian (with the geodesics which belong to it) turned round so as to occupy successively the positions of all the other meridians. The same remark applies of course to the projections; the figure shows the projections cutting at right angles a given radius of the circle; and this radius (with the projections belonging to it) is then to be turned round so as to occupy successively the positions of all the other radii. We may imagine the several geodesics turned round separately, each through a different angle, so as to bring each of them to pass through one and the same point of the surface; we have then the geodesics drawn in all directions through this point of the surface; doing the same thing with the projections, we have, it is clear, the projections of the geodesics drawn in all directions through the point. It is easy, by drawing the projections each on a separate circle of paper, and passing a pin through the centres, to form a model by means of which an accurate figure of the projection may be constructed. But I content myself with a mere diagram (fig. 4).

Fig. 4.
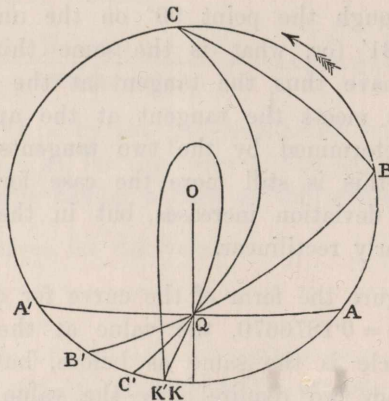
25. Taking a point Q so low down on the surface that the geodesic at right angles to the meridian through Q is a simple arc $A'A$, then imagine the two extremities A, A' each moving in the same sense round the circle, but A faster than A', so as to assume the positions B, B'; C, C'; and so on to K, K' coinciding with each other. We have the arcs $B'B$, $C'C$, and so on until we come to the loop form $K'K$: after which we have L' in advance of L, and so on to curves of any number of convolutions. Considering any two arcs—$B'B$, $C'C$—and drawing the geodesic BC which joins their extremities B and C, then any geodesic through Q intermediate to $B'B$, $C'C$, or, say, to QB, QC, will meet the arc BC; while the geodesics through Q extramediate to QB, QC *will not meet, or will only after a convolution or convolutions meet,*

the arc BC. This of course corresponds to the Lobatschewskian theory, according to which we have through a point Q to the extremities at infinity of a line BC, two distinct lines QB, QC, said to be the parallels through Q of the line BC; and which are such that any line through Q intermediate to QB, QC meets the line BC; while any line through Q extramediate to QB, QC *does not meet* the line BC.

26. It is interesting to connect the theory of the geodesics of the pseudosphere with the general theory of geodesics. Starting with the form

$$ds^2 = \cot^2 \theta \, d\theta^2 + \sin^2 \theta \, d\phi^2, \quad = E \, d\theta^2 + 2F \, d\theta \, d\phi + G \, d\phi^2,$$

we have $E = \cot^2 \theta$, $F = 0$, $G = \sin^2 \theta$; and therefore $E + 1 = \dfrac{1}{G}$, or $E = \dfrac{1}{G} - 1$, and the differential equation of the geodesic becomes

$$E\theta' \, . \, 2G_1 \theta' \phi' - G\phi' (E_1 \theta'^2 - G_1 \phi'^2) + 2EG \, (\theta' \phi'' - \theta'' \phi') = 0,$$

that is,

$$\phi' \left[(2EG_1 - GE_1) \theta'^2 + GG_1 \phi'^2 \right] + 2EG \, (\theta' \phi'' - \theta'' \phi') = 0,$$

where

$$E_1 = \frac{dE}{d\theta}, \quad G_1 = \frac{dG}{d\theta};$$

and writing here

$$E = \frac{1}{G} - 1,$$

we have

$$E_1 = -\frac{G_1}{G^2}, \quad 2EG_1 - E_1 G = G_1 \left(\frac{3}{G} - 2 \right).$$

Moreover, from $G = \sin^2 \theta$, we find $G_1 = 2 \sqrt{G \, . \, 1 - G}$; and the equation becomes

$$\frac{\sqrt{G}}{\sqrt{1 - G}} \left[\left(\frac{3}{G} - 2 \right) \theta'^2 + G\phi'^2 \right] \phi' + \theta' \phi'' - \theta'' \phi' = 0.$$

Introducing here G in place of θ by the equation $G = \sin^2 \theta$, we have

$$\theta' = \frac{G'}{2 \sqrt{G \, . \, 1 - G}},$$

$$\theta'' = \frac{1}{4G^{\frac{3}{2}} (1 - G)^{\frac{3}{2}}} \{ 2G (1 - G) G'' - G'^2 (1 - 2G) \},$$

and the equation thus becomes

$$(3 - 2G) \, G'^2 \phi' + 4G^3 (1 - G) \, \phi'^3 + 2G (1 - G) \, G' \phi'' + \{ -2G (1 - G) \, G'' + (1 - 2G) \, G'^2 \} \, \phi' = 0.$$

The whole term in ϕ' is thus $\phi' \{ -2G (1 - G) \, G'' + (4 - 4G) \, G'^2 \}$, which divides by $2 (1 - G)$; the whole equation thus divides by $2 (1 - G)$, and omitting this factor, the equation becomes

$$\phi' (2G'^2 - GG'') + 2\phi'^3 G^3 + \phi'' GG' = 0,$$

which is simplified by introducing $H = \sin^2 \theta = \dfrac{1}{G}$, instead of G. We, in fact, have

$$G' = \frac{-H'}{H^2}, \quad G'' = \frac{-H''}{H^2} + \frac{2H'^2}{H^3},$$

and substituting these values, the equation becomes

$$\phi' \left(\frac{2H'^2}{H^4} + \frac{H''}{H^3} - \frac{2H'^2}{H^4} \right) + \frac{2}{H^3} \phi'^3 - \frac{H'}{H^3} \phi'' = 0;$$

viz. this is

$$H'' \phi' + 2\phi'^3 - H' \phi'' = 0.$$

Writing herein $\phi + \alpha = \sqrt{K}$ (α an arbitrary constant), we have

$$\phi' = \frac{\frac{1}{2}K'}{\sqrt{K}}, \quad \phi'' = \frac{\frac{1}{2}K''}{\sqrt{K}} - \frac{\frac{1}{4}K'^2}{K\sqrt{K}},$$

and the equation becomes

$$H'' \frac{\frac{1}{2}K'}{\sqrt{K}} + \frac{\frac{1}{4}K'^3}{K\sqrt{K}} - \frac{\frac{1}{2}H'K''}{\sqrt{K}} + \frac{\frac{1}{4}H'K'^2}{K\sqrt{K}} = 0;$$

viz. this is

$$2(H''K' - H'K'')K + (K' + H')K'^2 = 0,$$

which is satisfied by $K' + H' = 0$ or $K + H = \beta$, β an arbitrary constant. Substituting for K, H their values, this is

$$(\phi + \alpha)^2 + \sin^2 \theta = \beta,$$

that is,

$$\alpha^2 - \beta + (\phi^2 + \operatorname{cosec}^2 \theta) + 2\alpha\phi = 0,$$

or, what is the same thing,

$$A + B(\phi^2 + \operatorname{cosec}^2 \theta) + C\phi = 0,$$

where the ratios $A : B : C$ are arbitrary.