836.

ON THE QUATERNION EQUATION $qQ - Qq' = 0$.

[From the *Messenger of Mathematics*, vol. XIV. (1885), pp. 108—112.]

I CONSIDER the equation $qQ - Qq' = 0$, where q, q' are given quaternions, and Q is a quaternion to be determined. Obviously a condition must be satisfied by the given quaternions; for, substituting in the given equation for q, q', Q their values, say $w + ix + jy + kz$, $w' + ix' + jy' + kz'$, and $W + iX + jY + kZ$ respectively, and equating to zero the scalar part and the coefficients of i, j, k, we have four equations linear in W, X, Y, Z, and then eliminating these quantities, we have the condition in question. Supposing the condition satisfied, the ratios of W, X, Y, Z are then completely determined, and the required quaternion Q is thus determinate except as to a scalar factor, or say Q is = product of an arbitrary scalar into a determinate quaternion expression.

It might, at first sight, appear that the condition is that the given quaternions shall have their tensors equal, $Tq = Tq'$; for the equation gives $Tq \,.\, TQ - TQ \,.\, Tq' = 0$, that is, $TQ\,(Tq - Tq') = 0$. But we cannot thence infer, and it is not true, that the condition is $Tq - Tq' = 0$; the formula does not give the required condition at all, but the conclusion to which it leads is that, when the condition is satisfied, then in general (that is, unless $Tq - Tq' = 0$) the required quaternion is an imaginary quaternion (or, as Hamilton calls it, a biquaternion) having its tensor $TQ = 0$. In the particular case where the given quaternions are such that $Tq - Tq' = 0$, then the required quaternion Q is determined less definitely, viz. it becomes = product of an arbitrary scalar into a not completely determined quaternion expression; and it is thus in general such that TQ is not $= 0$. In explanation, observe that, for the particular case in question, the four linear equations for W, X, Y, Z reduce themselves to two independent relations, and they give therefore for the ratios of W, X, Y, Z expressions involving an arbitrary parameter A; these expressions cannot, it is clear, be deduced from the determinate expressions which belong to the general case.

Instead of directly working out the condition in the manner indicated above, I present the investigation in a synthetic form as follows:

Taking v, v' for the vector parts of the two given quaternions, so that $q = w + v$, $q' = w' + v'$, I write for shortness

$$\begin{aligned}
\theta &= w - w', \\
\alpha &= v^2 + v'^2, \quad = -x^2 - y^2 - z^2 - x'^2 - y'^2 - z'^2, \\
\beta &= v^2 - v'^2, \quad = -x^2 - y^2 - z^2 + x'^2 + y'^2 + z'^2, \\
D &= -\theta(\alpha - \theta^2), \\
A &= \beta - \theta^2, \\
B &= \beta + \theta^2;
\end{aligned}$$

so that θ, α, β, D, A, B are all of them scalars. With these I form a quaternion $Q = (D + Av)(D + Bv')$; I say that we have identically

$$qQ - Qq' = \{D - v \cdot v'^2 + v' \cdot v^2 + vv' \cdot \theta\}(\theta^4 - 2\alpha\theta^2 + \beta^2).$$

It of course follows that, if $\theta^4 - 2\alpha\theta^2 + \beta^2 = 0$, then $qQ - Qq' = 0$, viz. $\theta^4 - 2\alpha\theta^2 + \beta^2 = 0$ is the condition $\Omega = 0$, for the existence of the required quaternion; and this condition being satisfied, then (omitting the arbitrary scalar factor) the value of the quaternion is $Q = (D + Av)(D + Bv')$, a value giving $T(Q) = 0$, that is, $W^2 + X^2 + Y^2 + Z^2 = 0$. If $\theta = 0$ (that is, $w = w'$), then the condition becomes $\beta = 0$, that is, $x^2 + y^2 + z^2 - x'^2 - y'^2 - z'^2 = 0$; and these two conditions being satisfied, Q ceases to have the determinate value given by the foregoing formula: it has a value involving an arbitrary parameter, and is no longer such that $W^2 + X^2 + Y^2 + Z^2 = 0$.

The identical equation is at once verified; we have

$$\begin{aligned}
qQ - Qq' &= (w + v)Q - Q(w' - v') \\
&= \theta\ (D^2 + DAv + DBv' + ABvv') \\
&\quad + v\ (D^2 + DAv + DBv' + ABvv') \\
&\quad - \ (D^2 + DAv + DBv' + ABvv')\,v' \\
&= \theta D^2 + DAv^2 - DBv'^2 \\
&\quad + v\ (DA\theta + D^2 - ABv'^2) \\
&\quad + v'\ (DB\theta + ABv^2 - D^2) \\
&\quad + vv'\,(\theta AB + DB - DA).
\end{aligned}$$

The first line is here $= D\{D\theta + Av^2 + Bv'^2\}$, viz. the term in $\{\ \}$ is

$$\begin{aligned}
&-(\alpha - \theta^2)\theta^2 + (\beta - \theta^2)v^2 - (\beta + \theta^2)v'^2, \\
&= -\alpha\theta^2 + \theta^4 + \beta \cdot \beta - \theta^2 \cdot \alpha, \\
&= \theta^4 - 2\alpha\theta^2 + \beta^2;
\end{aligned}$$

and similarly each of the other lines contains the factor $\theta^4 - 2\alpha\theta^2 + \beta^2$, and the equation is thus seen to hold good.

The tensor of Q is $=(D^2 - A^2v^2)(D^2 - B^2v'^2)$; and we have

$$D^2 - A^2v^2 = (\alpha - \theta^2)^2\,\theta^2 - v^2(\beta - \theta^2),$$
$$= \theta^6 - (2\alpha + v^2)\,\theta^4 + (\alpha^2 + 2\beta v^2)\,\theta^2 - v^2\beta^2,$$

which, observing that

$$\alpha^2 + 2\beta v^2 = \beta^2 + 2\alpha v^2 \quad \text{is} \quad = (\theta^2 - v^2)(\theta^4 - 2\alpha\theta^2 + \beta^2);$$

and similarly

$$D^2 - B^2v'^2 \qquad \text{is} \quad = (\theta^2 - v'^2)(\theta^4 - 2\alpha\theta^2 + \beta^2);$$

hence the tensor is

$$TQ = (\theta^2 - v^2)(\theta^2 - v'^2)(\theta^4 - 2\alpha\theta^2 + \beta^2)^2,$$

which, in virtue of $\theta^4 - 2\alpha\theta^2 + \beta^2 = 0$, is $= 0$.

The particular case is when $Tq - Tq' = 0$, that is, $w^2 - v^2 - w'^2 + v'^2 = 0$, or say $w^2 - w'^2 = v^2 - v'^2$, that is, $\theta(w + w') = \beta$. Combining with this the general condition $\theta^4 - 2\alpha\theta^2 + \beta^2 = 0$, we find $\theta^2\{\theta^2 - 2\alpha + (w + w')^2\} = 0$, or in the second factor, for θ and α substituting their values, we have $2\theta^2(w^2 - v^2 + w'^2 - v'^2) = 0$, that is, $2\theta^2(Tq + Tq') = 0$. Attending to the assumed relation $Tq - Tq' = 0$, the second factor can only vanish if $Tq = 0$, $Tq' = 0$; hence, disregarding this more special case, the factor which vanishes must be the first factor, that is, $\theta = 0$; or the equations $Tq - Tq' = 0$ and $\theta^4 - 2\alpha\theta^2 + \beta^2 = 0$ give $\beta = 0$ and $\theta = 0$, that is, we have as already mentioned $w - w' = 0$, and $x^2 + y^2 + z^2 = x'^2 + y'^2 + z'^2$, viz. the given quaternions have their scalars equal, and the squares of their vectors also equal. The equation here is $vQ - Qv' = 0$; and writing $v^2 = v'^2 = -p^2$, we see at once that a solution is

$$Q = (-p^2 + vv') + A\,(v + v'),$$

where A is an arbitrary scalar; in fact, with this value of Q, we have at once vQ and Qv' each

$$= -p^2(v + v') + A\,(-p^2 + vv');$$

and the equation $vQ - Qv' = 0$ is thus satisfied. The value of the tensor is easily found to be

$$TQ = 2\,(A^2 + p^2)(p^2 + xx' + yy' + zz'),$$

which is not $= 0$.

In accordance with a remark in the introductory paragraphs, the solution

$$Q = -p^2 + vv' + A\,(v + v')$$

is not comprised in the general solution. As to this, observe that, in the case in question $\theta = 0$, $\beta = 0$, we have from the general theorem the form

$$Q = \left(-\frac{\alpha\theta}{\beta} + v\right)\left(-\frac{\alpha\theta}{\beta} + v'\right);$$

that is,

$$Q = \frac{\alpha^2\theta^2}{\beta^2} + vv' - \frac{\alpha\theta}{\beta}(v + v');$$

in the condition $\theta^4 - 2\alpha\theta^2 + \beta^2 = 0$, writing $\theta = 0$, we have $\frac{\beta^2}{\theta^2} = 2\alpha$, or for α writing its value $= -2p^2$, we have $\alpha^2 = 4p^4$, $\frac{\theta^2}{\beta^2} = -\frac{1}{4p^2}$, and thence $\frac{\alpha^2\theta^2}{\beta^2} = -p^2$, and $\frac{\alpha\theta}{\beta} = \lambda p$, if λ denote the $\sqrt{(-1)}$ of ordinary algebra. The resulting formula is thus

$$Q = -p^2 + vv' - \lambda p\,(v + v'),$$

which corresponds to the determinate value $-\lambda p$ of the constant A.

The foregoing solution $Q = -p^2 + vv' + A\,(v + v')$ may be easily identified with that given pp. 124, 125 of Tait's *Elementary Treatise on Quaternions*; the case there considered is that of a real quaternion, and it was therefore assumed that the two conditions $w = w'$, and $x^2 + y^2 + z^2 = x'^2 + y'^2 + z'^2$, were each of them satisfied.

The theory of quaternions is, as is well known, identical with that of matrices of the second order; the identity is, in effect, established by the remark and footnote "Linear Associative Algebra," *American Journal of Mathematics*, t. IV. (1881), p. 132. Writing x, y, z, w for Peirce's imaginaries i, j, k, l, these have the multiplication table

	x	y	z	w
x	x	y	0	0
y	0	0	x	y
z	z	w	0	0
w	0	0	z	w

Then if λ, $= \sqrt{(-1)}$, be the imaginary of ordinary algebra, and i, j, k the quaternion imaginaries, the relations between i, j, k and x, y, z, w are

$$\begin{aligned} x &= \tfrac{1}{2}(\ \ 1 - \lambda i), &\text{or conversely}\quad 1 &= \ \ x + w, \\ y &= \tfrac{1}{2}(\ \ j - \lambda k), &\text{,,}\qquad i &= \lambda\,(x - w), \\ z &= \tfrac{1}{2}(-j - \lambda k), &\text{,,}\qquad j &= \ \ (y - z), \\ w &= \tfrac{1}{2}(\ \ 1 + \lambda i), &\text{,,}\qquad k &= \lambda\,(y + z); \end{aligned}$$

and we can thus at once express a quaternion as a linear function of the x, y, z, w, or a linear function of the x, y, z, w as a quaternion. And we then consider

$ax + by + cz + dw$ as denoting the matrix $\begin{vmatrix} a, & b \\ c, & d \end{vmatrix}$, we obtain for the product of two matrices the ordinary formula

$$\begin{vmatrix} a, & b \\ c, & d \end{vmatrix} \cdot \begin{vmatrix} a', & b' \\ c', & d' \end{vmatrix} = \begin{matrix} (a, b) \\ (c, d) \end{matrix} \begin{vmatrix} \overline{(a', c'),} & \overline{(b', d')} \\ \text{,,} & \text{,,} \\ \text{,,} & \text{,,} \end{vmatrix};$$

viz. we have

$$(ax + by + cz + dw)(a'x + b'y + c'z + d'w) = aa'x^2 + bc'yz + \&c.,$$
$$= (aa' + bc')\,x + (ab' + bd')\,y + (ca' + dc')\,z + (cb' + dd')\,w,$$

in accordance with the formula for the product of the two matrices. Observe that, writing

$$A + Bi + Cj + Dk = A\,(x + w) + B\lambda\,(x - w) + C\,(y - z) + D\lambda\,(y + z)$$
$$= ax + by + cz + dw,$$

we have

$$a,\ d,\ b,\ c = A + B\lambda,\quad A - B\lambda,\quad C + D\lambda,\quad -C + D\lambda,$$

and thence

$$ad - bc = A^2 + B^2 + C^2 + D^2,$$

so that the determinant of the matrix corresponds to the tensor of the quaternion.