837.

ON THE SO-CALLED D'ALEMBERT CARNOT GEOMETRICAL PARADOX.

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THE present note has reference to Prof. Sylvester's paper on this subject [*l. c.*, pp. 92—96]. I cannot admit that D'Alembert and Carnot raised a well-founded objection "to the then and even now too prevalent interpretation of the meaning of the geometrical positive and negative": it appears to me that the objection was not a well-founded one.

Consider through the origin K an indefinite line $t'Kt$, and measure off from K in the sense Kt a distance equal to the positive quantity α, and let m be the extremity of the distance thus measured off. There is not in the ordinary theory any reason why the distance Km should be $= +\alpha$ rather than $= -\alpha$; it is $= +\alpha$, if Kt be the positive sense of the line through K, and it is $= -\alpha$ if Kt' be the positive sense of the line through K; if it be undetermined which of the two is the positive sense, then the distance Km is $= \pm\alpha$, the sign being essentially indeterminate.

The problem is from a point K outside a given circle to draw a line Kmm' such that the intercepted portion mm' within the circle has a given value c.

Supposing that the line from K to the centre meets the circle in the points A, B at the distances $KA = a$, $KB = b$; then if $Km = r$, we have $ab = r(c + r)$, or $r = -\frac{1}{2}c \pm \sqrt{(\frac{1}{4}c^2 + ab)}$; viz. we have for r, not simultaneously but alternatively, the positive value $-\frac{1}{2}c + \sqrt{(\frac{1}{4}c^2 + ab)}$, and the negative value $-\frac{1}{2}c - \sqrt{(\frac{1}{4}c^2 + ab)}$, the latter of these being the greatest in absolute magnitude; say the values are $+\rho_1$ and $-\rho_2$. We may with either of these values construct the point m; viz. we obtain m as one of the intersections of the given circle with the circle centre K and radius ρ_1, or else with the circle centre K and radius $-\rho_2$ (that is, radius ρ_2); and attending to the intersections on the same side of the line from K to the centre, it happens that the two points m thus determined are on one and the same line $t'Kt$; but there is no *à priori* reason why the positive senses should be the same, and they are in fact opposite to each other, in the two cases respectively; in the one case we measure off the distance ρ_1 in the sense Kt, in the other case the distance $-\rho_2$ in the sense Kt'; that is, we in fact measure off the positive distances $+\rho_1$, and $+\rho_2$, in one and the same sense Kt; thus obtaining for the point m one or the other extremity of a determinate secant through K.

The best illustration is I think in the elementary problem of finding the perpendicular distance of a given line from the origin. Let $Ax+By+C=0$ be the equation of the given line: and first let a line be drawn *in a determinate sense*, say at the inclination θ to the positive part of the axis of x, to meet the given line. Taking r for the distance from the origin of the point of intersection, we have, for the coordinates of the point of intersection, $x, y = r\cos\theta, r\sin\theta$; and thence

$$r(A\cos\theta + B\sin\theta) + C = 0,$$

that is,

$$r = \frac{-C}{A\cos\theta + B\sin\theta},$$

a perfectly determinate value. But the perpendicular on the given line may be considered as drawn in one or the other of two opposite senses; that is, we have at pleasure

$$\cos\theta,\ \sin\theta = \frac{A}{\sqrt{(A^2+B^2)}},\ \frac{B}{\sqrt{(A^2+B^2)}},$$

or else

$$= \frac{-A}{\sqrt{(A^2+B^2)}},\ \frac{-B}{\sqrt{(A^2+B^2)}};$$

and thence $r = \dfrac{-C}{\sqrt{(A^2+B^2)}}$, or else $r = \dfrac{+C}{\sqrt{(A^2+B^2)}}$; that is, the perpendicular distance is $= \dfrac{\pm C}{\sqrt{(A^2+B^2)}}$, with the essentially indeterminate sign $\pm$, because the distance may be considered as drawn from the origin in one or the other of the two opposite senses.