838.

ON THE TWISTED CUBICS UPON A QUADRIC SURFACE.

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A CUBIC (twisted cubic) on a quadric surface meets each generator of the one kind twice, and each generator of the other kind once (Salmon, *Solid Geometry*, Ed. 4, p. 301). There are thus on the surface two kinds of cubics, viz. distinguishing for convenience the generating lines as generators and directors, a cubic may meet each generator twice and each director once; or it may meet each generator once and each director twice. And two cubic curves are accordingly of the same kind or of different kinds.

Consider for example the quadric surface $xw - yz = 0$, and the cubic

$$x : y : z : w = 1 : \phi : \phi^2 : \phi^3.$$

If we call the line $(x = ky,\ kw = z)$ a generator, and the line $(x = kz,\ kw = y)$ a director, for the intersections with a generator we have $1 = k\phi$, $k\phi^3 = \phi^2$; equations with the common root $k\phi = 1$, or there is a single intersection; for the intersections with a director, $1 = k\phi^2$, $k\phi^3 = \phi$, equations with the common roots $k\phi^2 = 1$, or there are two intersections.

Consider on the same quadric surface the cubic

$$x : y : z : w = \theta-\alpha.\theta-\beta.\theta-\gamma : \theta-\alpha.\theta-\epsilon.\theta-\zeta : \theta-\delta.\theta-\beta.\theta-\gamma : \theta-\delta.\theta-\epsilon.\theta-\zeta;$$

or as, for shortness, I write these,

$$x : y : z : w = \alpha\beta\gamma : \alpha\epsilon\zeta : \delta\beta\gamma : \delta\epsilon\zeta,$$

viz. α is written to denote $\theta - \alpha$, and so in other cases; for the intersections with a generator, we have $\alpha\beta\gamma = k\alpha\epsilon\zeta$, $k\delta\epsilon\zeta = \delta\beta\gamma$, equations having the two common roots

$\beta\gamma = k\epsilon\zeta$; or there are two intersections with the generator. And in like manner there is one intersection with the director. The cubic

$$x : y : z : w = \alpha\beta\gamma : \alpha\epsilon\zeta : \delta\beta\gamma : \delta\epsilon\zeta$$

is thus a cubic of different kind from

$$x : y : z : w = 1 : \phi : \phi^2 : \phi^3.$$

And in the same manner it appears that the cubic

$$x : y : z : w = \alpha\beta\gamma : \delta\beta\gamma : \alpha\epsilon\zeta : \delta\epsilon\zeta$$

is of the same kind with

$$x : y : z : w = 1 : \phi : \phi^2 : \phi^3.$$

The two cubics of different kinds intersect in 5 points; the two cubics of the same kind in only 4 points. In fact, for the intersections with the cubic

$$x : y : z : w = 1 : \phi : \phi^2 : \phi^3,$$

we have $xz - y^2 = 0$, $yw - z^2 = 0$, and substituting herein the θ-values,

$$x : y : z : w = \alpha\beta\gamma : \alpha\epsilon\zeta : \delta\beta\gamma : \delta\epsilon\zeta,$$

we find

$$\alpha\delta\beta^2\gamma^2 - \alpha^2\epsilon^2\zeta^2 = 0, \quad \alpha\delta\epsilon^2\zeta^2 - \delta^2\beta^2\gamma^2 = 0,$$

equations with the common five roots $\delta\beta^2\gamma^2 - \alpha\epsilon^2\zeta^2 = 0$; whereas for the θ-values $x : y : z : w = \alpha\beta\gamma : \delta\beta\gamma : \alpha\epsilon\zeta : \delta\epsilon\zeta$, we have $\alpha^2\beta\gamma\epsilon\zeta - \delta^2\beta^2\gamma^2 = 0$, $\delta^2\beta\gamma\epsilon\zeta - \alpha^2\epsilon^2\zeta^2 = 0$, equations with the common four roots $\alpha^2\epsilon\zeta - \delta^2\beta\gamma = 0$.

I remark in passing that the equations just obtained have each of them a root $\theta = \infty$, but this would have been avoided if the θ-equations had been taken in the slightly altered form

$$x : y : z : w = \alpha\beta\gamma : b\alpha\epsilon\zeta : c\delta\beta\gamma : bc\delta\epsilon\zeta,$$

and

$$x : y : z : w = \alpha\beta\gamma : c\delta\beta\gamma : b\alpha\epsilon\zeta : bc\delta\epsilon\zeta,$$

b, c being arbitrary constants; and the special value is thus of no importance.

The two cubics intersecting in 5 points constitute the complete intersection of a quadric surface and a cubic surface; and conversely when a quadric surface and a cubic surface intersect in two cubics, then the cubics must have 5 intersections, and are thus by what precedes cubics of different kinds in regard to the quadric surface. In fact, considering two cubics meeting in 5 points, we can, through these 5 points, through 2 points assumed at pleasure on the first cubic, and through 2 points assumed at pleasure on the second cubic, draw a determinate quadric surface: this meets each of the two cubics in $5 + 2$ points, and consequently it entirely contains the cubic; viz. we have through the two cubics a determinate quadric surface. Again, through the 5 points, through 5 points at pleasure on the first cubic and through 5 points at pleasure on the second cubic, we may draw a cubic surface; this meets each cubic in $5+5$ points, and therefore it entirely contains the cubic; we have thus a cubic surface through the two cubics. The cubic surface has been subjected only to $5+5+5=15$ conditions, and the equation thus contains homogeneously $20-15, =5$ constants; viz. if $\Theta=0$ be the quadric surface through the two cubics the general form is $(\alpha x+\beta y+\gamma z+\delta w)\Theta+\lambda U=0$; disregarding the term in Θ, we have thus, in fact, a determinate cubic $U=0$.

Taking as above the first cubic as given by the equations

$$x : y : z : w = 1 : \phi : \phi^2 : \phi^3,$$

and the second cubic as given by the equations

$$x : y : z : w = \alpha\beta\gamma : \alpha\epsilon\zeta : \delta\beta\gamma : \delta\epsilon\zeta,$$

the equation of the cubic surface which contains the two cubics will be of the form

$$(Ax+By+Cz+Dw)(yw-z^2)+(A''x+B''y+C''z+D''w)(xz-y^2)=0.$$

Substituting herein the θ-values and omitting the common factor $\alpha\epsilon^2\zeta^2-\delta\beta^2\gamma^2$, we have

$$\delta(A\alpha\beta\gamma+B\alpha\epsilon\zeta+C\delta\beta\gamma+D\delta\epsilon\zeta)-\alpha(A''\alpha\beta\gamma+B''\alpha\epsilon\zeta+C''\delta\beta\gamma+D''\delta\epsilon\zeta)=0,$$

which is of the fourth order in θ. We may assume $C''=0$, and $D''=0$; in fact, the terms in C'' and D'' might be got rid of by the substitutions

$$z(xz-y^2)=y(xw-yz)-x(yw-z^2),$$
$$w(xz-y^2)=z(xw-yz)-y(yw-z^2);$$

we then have 6 coefficients A, B, C, D, A'', B'', and equating to zero the terms in $\theta^0, \theta^1, \theta^2, \theta^3, \theta^4$, the ratios of these are determined by the five equations: and we have thus the required cubic surface.

Starting from either cubic considered as a curve on the given quadric surface; if through the centre we describe a cubic surface, this meets the quadric surface in a second cubic and the two cubics intersect each other in 5 points. In particular, if the one cubic is $x : y : z : w=1 : \phi : \phi^2 : \phi^3$, it appears by what precedes that the other cubic may be taken to be

$$x : y : z : w=\alpha\beta\gamma : \alpha\epsilon\zeta : \delta\beta\gamma : \delta\epsilon\zeta,$$

intersecting the former in the 5 points given by the equation $\delta\beta^2\gamma^2-\alpha\epsilon^2\zeta^2=0$. The five points are of course points of contact of the quadric surface and the cubic surface.

A very simple instance is the following. The quadric surface $xw-yz=0$ and the cubic surface

$$y(xz-y^2)+z(yw-z^2)=0$$

intersect in the two cubics

$$x : y : z : w=1 : \phi : \phi^2 : \phi^3 \text{ and } x : y : z : w=1 : \theta^2 : \theta : \theta^3.$$

The two cubics meet in the 5 points

$$\theta = (0,\ \infty,\ 1,\ \omega,\ \omega^2);\quad \phi = (0,\ \infty,\ 1,\ \omega^2,\ \omega),$$

or, what is the same thing, in the 5 points

$$(x,\ y,\ z,\ w) = (1,\ 0,\ 0,\ 0),\ \ (0,\ 0,\ 0,\ 1),\ \ (1,\ 1,\ 1,\ 1),\ \ (1,\ \omega,\ \omega^2,\ 1),\ \ (1,\ \omega^2,\ \omega,\ 1),$$

where ω is an imaginary cube root of unity.

I was anxious to work out this result not for its own sake, but as an illustration of a point which first arises as to octics: a unicursal octic curve is not in general situate on any surface lower than a quartic surface; if on the octic we take at pleasure 33 points, we may through these draw two quartic surfaces $U=0$, $V=0$, each of which will entirely contain the curve: the two surfaces meet in a second octic also unicursal, and the two curves intersect in 34 points, which are points of contact of the two quartic surfaces. We cannot, by adjoining to the given octic any curve of an inferior to its own, obtain a complete intersection of two surfaces: the most simple case is when we adjoin to the given octic as above another unicursal octic, and thus obtain a complete intersection of two quartic surfaces. It is to be observed that the second curve is determined completely and uniquely by the given curve: considering the former curve as given by the expressions of the coordinates in terms of a parameter ϕ, the determination of the like expressions for the latter curve would appear to be a problem of very great difficulty.