845.

ON THE ORTHOMORPHOSIS OF THE CIRCLE INTO THE PARABOLA.

[From the *Quarterly Journal of Pure and Applied Mathematics*, vol. XX. (1885), pp. 213—220.]

It is remarked by Schwarz (see his Memoir "Ueber einige Abbildungsaufgaben," *Crelle*, t. LXX. (1869), pp. 105—120; p. 115), that the circle $x'^2 + y'^2 - 1 = 0$ can be orthomorphosed into the parabola $y^2 = 4(1 - x)$ by the equation

$$\sqrt{(x' + iy')} = \tan \tfrac{1}{4}\pi \sqrt{(x + iy)}:$$

viz. this equation establishes a (1, 1) relation between the points interior to the circle and those interior to the parabola, which, quà relation between $x' + iy'$ and $x + iy$, will be orthomorphic, that is, infinitesimal elements of the one area will correspond to similar infinitesimal elements of the other area. The diameter $y' = 0$ of the circle is transformed into the axis $y = 0$ of the parabola, and figs. 1, 2 are symmetrical on the two sides of these lines respectively; we may therefore consider only the transformation

Fig. 1.

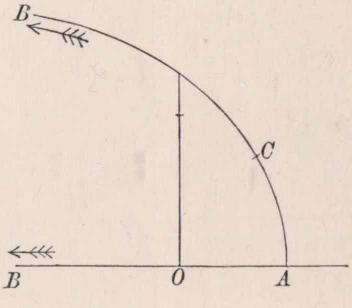
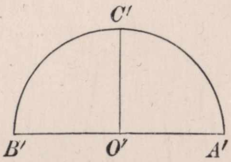
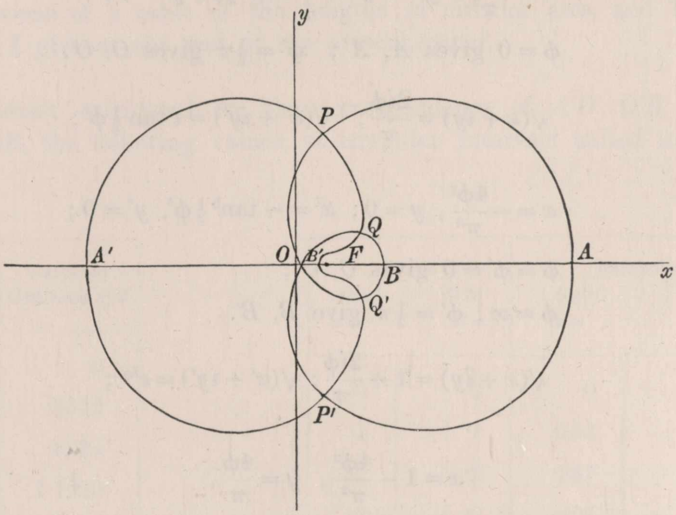


Fig. 2.



of the upper semicircle into the upper half of the parabola; we have (see figs. 1 and 2) A' corresponding to the vertex A, O' to the focus O, and B' to the point at infinity ($x = -\infty$) on the axis of the parabola; the semicircular arc $A'C'B'$ corresponds to the infinite half-arc ACB of the parabola, the highest point C' corresponding to a point C between the vertex and the semi-latus rectum.

Fig. 3.



We may divide the circle by concentric circles and by radii; the corresponding curves for the parabola will be ovals and radials from the focus, the curves of each system being transcendental curves. Or we may divide the parabola by means of two systems of confocal parabolas; the corresponding curves for the circle will be (see fig. 3) two systems of orthotomic limaçons, those of the one system having B' for a crunode, and those of the other system having B' for an acnode.

To show that the circle thus corresponds to the parabola, it is only necessary to write $\sqrt{(x+iy)} = 1 + qi$, that is, $x = 1 - q^2$, $y = 2q$, implying $y^2 = 4(1-x)$, or (x, y) is a point on the parabola; and we then obtain for $x' + iy'$ a value of the form $\cos\theta' + i\sin\theta'$, that is, (x', y') is a point on the circle $x'^2 + y'^2 = 1$; but in reference to what follows, I give the proof in a somewhat more artificial form.

Writing log tan to denote the hyperbolic logarithm of the tangent, then if ϕ, ϕ' are such that

$$\phi = \log\tan\left(\tfrac{1}{4}\pi + \tfrac{1}{2}\phi'\right),$$

this equation, as is known, may also be presented in the forms

$$i\phi' = \log\tan\left(\tfrac{1}{4}\pi + \tfrac{1}{2}i\phi\right),$$

$$i\tan\tfrac{1}{2}\phi' = \tan\tfrac{1}{2}i\phi,$$

or, what is the same thing,

$$\tan\tfrac{1}{2}\phi' = \tanh\tfrac{1}{2}\phi,$$

where observe that, as ϕ' increases from 0 to $\frac{1}{2}\pi$, ϕ increases from 0 to ∞, and that always $\phi > \phi'$.

We can now establish as follows the three correspondences AO with $A'O'$, OB with $O'B'$, and ACB with $A'C'B'$.

(1) $$\sqrt{(x+iy)} = \frac{2\phi}{\pi};\quad \sqrt{(x'+iy')} = \tan\tfrac{1}{2}\phi';$$

that is,

$$x = \frac{4\phi^2}{\pi^2},\ y = 0;\quad x' = \tan^2\tfrac{1}{2}\phi',\ y' = 0;$$

$$\phi = 0 \text{ gives } A,\ A';\quad \phi' = \tfrac{1}{2}\pi \text{ gives } O,\ O'.$$

(2) $$\sqrt{(x+iy)} = \frac{2i\phi}{\pi};\quad \sqrt{(x'+iy')} = i\tan\tfrac{1}{2}\phi';$$

that is,

$$x = -\frac{4\phi^2}{\pi^2},\ y = 0;\ x' = -\tan^2\tfrac{1}{2}\phi',\ y' = 0;$$

$$\phi = \phi' = 0 \text{ gives } O,\ O';$$

$$\phi = \infty,\ \phi' = \tfrac{1}{2}\pi \text{ give } B,\ B'.$$

(3) $$\sqrt{(x+iy)} = 1 + \frac{2i\phi}{\pi};\ \sqrt{(x'+iy')} = e^{i\phi'};$$

that is,

$$x = 1 - \frac{4\phi^2}{\pi^2},\ y = \frac{4\phi}{\pi},$$

and therefore

$$y^2 = 4(1-x),$$

the parabola; and

$$x' = \cos 2\phi',\ y' = \sin 2\phi',$$

and therefore

$$x'^2 + y'^2 = 1,$$

the circle;

$$\phi = \phi' = 0 \text{ gives } A,\ A';\quad \phi = \infty,\ \phi' = \tfrac{1}{2}\pi \text{ give } B,\ B'.$$

Observe that for points in $O'A'$, $O'B'$, equidistant from O', we have $x' = \tan^2\frac{1}{2}\phi'$, $x' = -\tan^2\frac{1}{2}\phi'$; and corresponding hereto we have points in OA, OB on the axis of the parabola, the values of x being $x = \frac{4\phi'^2}{\pi^2}$, $x = -\frac{4\phi^2}{\pi^2}$, viz. the negative value is always the greater.

Observe further that, to the points $(x', y') = (\cos 2\phi', \sin 2\phi')$ and $(x', y') = (-\tan^2\frac{1}{2}\phi', 0)$ on the circle, and on the radius OB', correspond the points

$$(x,\ y) = \left(1 - \frac{4\phi^2}{\pi^2},\ \frac{2\phi}{\pi}\right),\ \text{ and } (x,\ y) = \left(-\frac{4\phi^2}{\pi^2},\ 0\right),$$

on the parabola and on the axis OB respectively; the axial distance of these two points is $\left(1 - \frac{4\phi^2}{\pi^2}\right) + \frac{4\phi^2}{\pi^2} = 1$, the radius of the circle, or the distance OA of the vertex and focus of the parabola; this is a rather curious theorem.

The function log tan $(45° + \frac{1}{2}\text{arg.})$ is tabulated, Legendre, *Théorie des Fonctions Elliptiques*, t. II. table iv. pp. 256—259, viz. writing as above ϕ' for the argument, we have hereby the value of ϕ, $= \log\tan(\frac{1}{4}\pi + \frac{1}{2}\phi')$, for every value of ϕ' from 0° to 90° at intervals of 30′ and to 12 decimals, and with fifth differences. Observe that ϕ' is thus given in degrees and minutes as a circular arc, ϕ as a number; it is convenient to have ϕ and ϕ' each as arcs, or each as numbers, the conversion being of course at once made by means of a table of the lengths of circular arcs, and I have calculated the Table which I give at the end of the present paper.

I had previously calculated for the correspondence of $A'O'$, $O'B'$ and $A'C'B'$ with AO, OB and ACB, the following values, at irregular intervals suited to the construction of a figure.

Circle $\angle A'O'P'$	Parabola Ordinate PM
0°	0
30	·3372
60	·6994
90	1·1220
120	1·6768
150	2·5817
160	3·1018
170	3·9869
172	4·2713
174	4·6378
176	5·1542
178	6·0258
179	6·9195
180°	∞

Circle		Parabola	
$O'A'$ $x'=$	$O'B'$ $x'=$	OA $x=$	OB $x=$
0	0	0	0
·1	− ·1	·152	− ·173
·2	− ·2	·287	− ·375
·3	− ·3	·407	− ·611
·4	− ·4	·515	− ·943
·5	− ·5	·615	− 1·257
·6	− ·6	·704	− 1·724
·7	− ·7	·787	− 2·368
·8	− ·8	·853	− 3·386
·9	− ·9	·935	− 5·368
1·0	− 1·0	1·000	− ∞

Write in general

$$\sqrt{(x+iy)} = p + qi, \ = \frac{2}{\pi}(\psi + i\phi);$$

viz. considering x, y as given, find thence p, q by the equations $x = p^2 - q^2$, $y = 2pq$: and then writing $\frac{1}{2}\pi p = \psi$, $\frac{1}{2}\pi q = \phi$, we have

$$\sqrt{(x' + iy')} = \tan\tfrac{1}{4}\pi(p + qi) = \tan(\tfrac{1}{2}\psi + \tfrac{1}{2}i\phi) = \frac{P + iQ}{1 - iPQ},$$

where

$$P = \tan \tfrac{1}{4}\pi p, \quad = \tan \tfrac{1}{2}\psi,$$

$$Q = \frac{1}{i} \tan \tfrac{1}{4}\pi q i, \quad = \tanh \tfrac{1}{4}\pi q, \quad = \frac{1}{i} \tan \tfrac{1}{2}\phi i, \quad = \tanh \tfrac{1}{2}\phi ;$$

whence, if we introduce ϕ' connected with ϕ by the equation $\phi = \log \tan (\tfrac{1}{4}\pi + \tfrac{1}{2}\phi')$ as before, we have $Q = \tan \tfrac{1}{2}\phi'$, and the formula is

$$\sqrt{(x' + iy')} = \frac{P + iQ}{1 - iPQ}, \quad P = \tan \tfrac{1}{2}\psi, \quad Q = \tan \tfrac{1}{2}\phi',$$

giving the values of x', y'.

It is clear that we have

$$\sqrt{(x' + iy')} = \frac{P - iQ}{1 + iPQ},$$

and thence

$$\sqrt{(x'^2 + y'^2)} = \frac{P^2 + Q^2}{1 + P^2Q^2}.$$

Hence, to the circle $x'^2 + y'^2 = c'^2$, corresponds in the parabola the curve

$$P^2 + Q^2 = c' (1 + P^2Q^2),$$

where $p + iq = \sqrt{(x + iy)}$, $P = \tan \tfrac{1}{4}\pi p$, $Q = \tanh \tfrac{1}{4}\pi q$. This is a complicated transcendental equation, and I do not see any convenient way of tracing the curve. The set of curves satisfy the differential equation

$$\frac{P\,dP + Q\,dQ}{P^2 + Q^2} = \frac{PQ\,(Q\,dP + P\,dQ)}{1 + P^2Q^2},$$

that is,

$$P\,dP\,(1 - Q^4) + Q\,dQ\,(1 - P^4) = 0,$$

where dP, dQ are given in terms of dp, dq by the equations

$$\frac{dP}{1 + P^2} = \tfrac{1}{4}\pi\, dp, \quad \frac{dQ}{1 - Q^2} = \tfrac{1}{4}\pi\, dq.$$

We have $y = 2pq$, and thence at the highest points, or summits of the several curves, $q\,dp + p\,dq = 0$. Combining these equations, we have

$$dP \;:\; dQ = p\,(1 + P^2) \;:\; -q\,(1 - Q^2) ;$$

and thence

$$pP\,(1 + Q^2) - qQ\,(1 - P^2) = 0,$$

as an equation to the locus of the summits in question. If p and q are small, then putting for a moment $\tfrac{1}{4}\pi = m$ for shortness, we have

$$P = mp + \tfrac{1}{3}m^3p^3, \quad Q = mq - \tfrac{1}{3}m^3q^3,$$

and the equation becomes

$$p^2\,(1 + \tfrac{1}{3}m^2p^2)\,(1 + m^2q^2) - q^2\,(1 - \tfrac{1}{3}m^2q^2)\,(1 - m^2p^2) = 0,$$

that is,

$$p^2 - q^2 + \tfrac{1}{3}m^2(p^4 + 6p^2q^2 + q^4) = 0\,;$$

writing this in the form

$$p^2 - q^2 + \frac{\pi^2}{48}[(p^2 - q^2)^2 + 8p^2q^2] = 0,$$

we find $x + \frac{\pi^2}{48}(x^2 + 2y^2) = 0$, or say $y^2 = -\frac{24}{\pi^2}x$, as the locus of the summits in the neighbourhood of the focus O, viz. the summits lie all of them, as might have been expected, on the left-hand (or negative side) of the focus.

I have constructed for the parabola, to the scale $1 = 1\frac{1}{2}$ inch, as accurately as the data enable, the figure corresponding to the concentric circles and the radii of the circle.

Resuming the equation $\sqrt{(x + iy)} = p + iq$, that is, $x = p^2 - q^2$ and $y = 2pq$, we have (p = const.), the curves $y^2 = 4p^2(p^2 - x)$, and (q = const.), $y^2 = 4q^2(q^2 + x)$, which are two systems of confocal parabolas, cutting each other at right angles; the curves of the former set all of them interior to the given parabola, those of the latter set of course cutting it at right angles.

Corresponding hereto in the circle, writing for a moment

$$\sqrt{(x' + iy')} = p' + iq',$$

we have

$$p' + iq' = \frac{P + iQ}{1 - iPQ},$$

whence

$$p' = P(1 - q'Q), \quad q' = Q(1 + p'P)\,;$$

whence eliminating P and Q successively, we find

$$p'^2 + q'^2 - p'\left(P - \frac{1}{P}\right) - 1 = 0,$$

$$p'^2 + q'^2 - q'\left(Q + \frac{1}{Q}\right) + 1 = 0\,;$$

say, for shortness, these are $p'^2 + q'^2 - 2mp' + 1 = 0$, and $p'^2 + q'^2 - 2nq' - 1 = 0$. Introducing the polar coordinates r', θ', we have

$$x' + iy' = r'(\cos\theta' + i\sin\theta'),$$

and thence

$$p',\ q' = \sqrt{(r')}\cos\tfrac{1}{2}\theta', \quad \sqrt{(r')}\sin\tfrac{1}{2}\theta'\,;$$

the equations thus become

$$r' - 1 - 2m\sqrt{(r')}\cos\tfrac{1}{2}\theta' = 0,$$

$$r' + 1 - 2n\sqrt{(r')}\sin\tfrac{1}{2}\theta' = 0,$$

which belong to two limaçons having each of them B' for a node and O for a focus, and which of course cut each other at right angles; see fig. 4, which is a mere

Fig. 4.

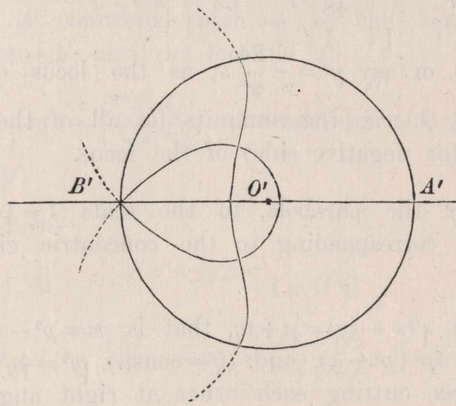


diagram. In fact, omitting for convenience the accents, but recollecting always that the curves belong to the figure of the circle, the first equation gives

$$(r-1)^2 = 4m^2 r \cos^2 \tfrac{1}{2}\theta', \quad = 2m^2 r\,(1+\cos\theta) = 2m^2\,(r+x),$$

that is,

$$(r^2+1-2m^2 x)^2 = 4\,(m^2+1)^2\, r^2.$$

Transforming to the point B' as origin, we must write $x+1$ for x, and then

$$r^2 = (x-1)^2 + y^2;$$

the equation thus becomes

$$\{x^2+y^2-2\,(m^2+1)\,x+2\,(m^2+1)\}^2 = 4\,(m^2+1)^2\,(x^2+y^2+2x+1),$$

that is,

$$\{x^2+y^2-2\,(m^2+1)\,x\}^2 \qquad = 4m^2\,(m^2+1)\,(x^2+y^2),$$

or say

$$(x^2+y^2)^2 - 4\,(m^2+1)\,(x^2+y^2)\,x + 4\,(m^2+1)\,(x^2-m^2y^2) = 0,$$

showing that the point B' is a crunode, the tangents there being $x = \pm\, my$.

Writing in the equation $y=0$, we have $x^2=0$, the crunode, and

$$x = 2\,\sqrt{(m^2+1)}\,\{\sqrt{(m^2+1)} \pm m\};$$

the product of these two values is $=4\,(m^2+1)$, that is, $=\left(P+\dfrac{1}{P}\right)^2$, viz. one value is greater, the other less, than 2. We see also that

$$2\,\sqrt{(m^2+1)}\,\{\sqrt{(m^2+1)} - m\},$$

the smaller root, is greater than 1. The curve corresponding to a parabola $y^2 = 4p^2(p^2 - x)$ is thus a crunodal limaçon, the crunode at B', and the loop lying wholly within the circle. Moreover, the loop includes within itself the centre O' of the circle.

The other curve is in like manner

$$\{r^2 + 1 + 2n^2x\}^2 = 4(n^2 - 1)r^2,$$

viz. transforming to the origin B', and therefore putting $x - 1$ for x, and

$$r^2 = (x-1)^2 + y^2,$$

the equation is

$$\{x^2 + y^2 + 2(n^2 - 1)x - 2(n^2 - 1)\}^2 = 4(n^2 - 1)^2(x^2 + y^2 - 2x + 1),$$

that is,

$$\{x^2 + y^2 + 2(n^2 - 1)x\}^2 = 4n^2(n^2 - 1)(x^2 + y^2),$$

or say

$$(x^2 + y^2)^2 + 4(n^2 - 1)(x^2 + y^2)x - 4(n^2 - 1)(x^2 + n^2y^2) = 0,$$

viz. the point B' is an acnode with the imaginary tangents $x = \pm iny$.

Writing in the equation $y = 0$, we have $x^2 = 0$ the acnode, and

$$x = -\sqrt{(n^2 - 1)}\{\sqrt{(n^2 - 1)} \pm n\},$$

where

$$\sqrt{(n^2 - 1)} = \tfrac{1}{2}\sqrt{\left(Q - \frac{1}{Q}\right)^2}$$

is real and may be taken to be positive; there is thus one root

$$-\sqrt{(n^2 - 1)}\{\sqrt{(n^2 - 1)} + n\},$$

which is negative; the other root, say

$$\sqrt{(n^2 - 1)}\{n - \sqrt{(n^2 - 1)}\},$$

is positive and less than 1; the curve is thus an acnodal limaçon, having B' for the acnode and cutting the axis outside the circle on the negative side of B', and inside the circle between B' and O'.

Taking B' as origin, the equation of the circle is $x^2 + y^2 - 2x = 0$, and we hence find for the intersection of the limaçon with the circle $n^2x = 2(n^2 - 1)$, that is, $x = 2\left(1 - \frac{1}{n^2}\right)$; whence also $y^2 = \frac{4}{n^2}\left(1 - \frac{1}{n^2}\right)$, values which belong to a real intersection; for $q = 0$, we have $Q = 0$, $n = \infty$, and therefore $(x, y) = (2, 0)$, viz. the intersection is at B'; for $q = \infty$, we have $Q = 1$, $n = 1$, and therefore $(x, y) = (0, 0)$, viz. the intersection is at A', results which are obviously right. Observe that for the other limaçon we have $x = 2\left(1 + \frac{1}{m^2}\right)$, $y = -\frac{4}{m^2}\left(1 + \frac{1}{m^2}\right)$, viz. there is no real intersection with the circle.

The Table above referred to.

$\phi = \log \tan (45^\circ + \frac{1}{2}\phi')$. $\phi = \log \tan (45^\circ + \frac{1}{2}\phi')$.

ϕ'		ϕ		ϕ'		ϕ	
0°	·0	·0	0°	46°	·8028515	·9062755	51°55′·546
1	0174533	0174542	1 0′·003	47	·8203047	·9316316	53 22 ·714
2	0349066	0349137	2 0 ·024	48	·8377580	·9574669	54 51 ·529
3	0523599	0523838	3 0 ·082	49	·8552113	·9838079	56 22 ·082
4	0698132	0698699	4 0 ·195	50	·8726646	1·0106832	57 54 ·473
5	0872665	0873774	5 0 ·381	51	·8901179	1·0381235	59 28 ·806
6	1047198	1049117	6 0 ·657	52	·9075712	1·0661617	61 5 ·194
7	1221730	1224781	7 1 ·152	53	·9250245	1·0948335	62 43 ·761
8	1396263	1400822	8 1 ·568	54	·9424778	1·1241772	64 24 ·637
9	1570796	1577296	9 2 ·269	55	·9509311	1·1542346	66 7 ·967
10	1745329	1754258	10 3 ·069	56	·9773844	1·1850507	67 53 ·904
11	1919862	1931766	11 4 ·092	57	·9948377	1·2166748	69 42 ·620
12	2094395	2109867	12 5 ·323	58	1·0122910	1·2491606	71 34 ·298
13	2268928	2288650	13 6 ·670	59	1·0297443	1·2825668	73 29 ·140
14	2443461	2468145	14 8 ·486	60	1·0471976	1·3169579	75 27 ·368
15	2617994	2648422	15 10 ·460	61	1·0646508	1·3524048	77 29 ·225
16	2792527	2829545	16 12 ·726	62	1·0821041	1·3889860	79 34 ·639
17	2967060	3011577	17 15 ·304	63	1·0995574	1·4267882	81 44 ·937
18	3141593	3194583	18 18 ·217	64	1·1170107	1·4659083	83 59 ·421
19	3316126	3378629	19 21 ·487	65	1·1344640	1·5064542	86 18 ·808
20	3490659	3563785	20 25 ·139	66	1·1519173	1·5485472	88 43 ·513
21	3665191	3750121	21 29 ·197	67	1·1693706	1·5923237	91 14 ·006
22	3839724	3937710	22 33 ·685	68	1·1868239	1·6379387	93 50 ·819
23	4014257	4126626	23 39 ·630	69	1·2042772	1·6855685	96 34 ·558
24	4188790	4316947	24 44 ·057	70	1·2217305	1·7354152	99 25 ·918
25	4363323	4508753	25 49 ·995	71	1·2391838	1·7877120	102 25 ·701
26	4537856	4702127	26 56 ·472	72	1·2566371	1·8427300	105 34 ·839
27	4712389	4897154	28 3 ·518	73	1·2740904	1·9007867	108 54 ·423
28	4886922	5093923	29 11 ·162	74	1·2915436	1·9622572	112 25 ·744
29	5061455	5292527	30 19 ·437	75	1·3089969	2·0275894	116 10 ·339
30	5235988	5493061	31 28 ·375	76	1·3264502	2·0973240	120 10 ·069
31	5410521	5695627	32 38 ·012	77	1·3439035	2·1721218	124 27 ·205
32	5585054	5900329	33 48 ·383	78	1·3613568	2·2528027	129 4 ·566
33	5759587	6107275	34 59 ·527	79	1·3788101	2·3404007	134 5 ·705
34	5934119	6316581	36 11 ·480	80	1·3962634	2·4362460	139 35 ·197
35	6108652	6528366	37 24 ·287	81	1·4137167	2·5420904	145 39 ·063
36	6283185	6742755	38 37 ·988	82	1·4311700	2·6603061	152 25 ·459
37	6457718	6959880	39 52 ·631	83	1·4486233	2·7942190	160 5 ·818
38	6632251	7179880	41 8 ·261	84	1·4660766	2·9487002	168 56 ·885
39	6806784	7402901	42 24 ·930	85	1·4835299	3·1313013	179 24 ·621
40	6981317	7629095	43 42 ·690	86	1·5009832	3·3546735	192 12 ·518
41	7155850	7858630	45 1 ·597	87	1·5184364	3·6425334	208 42 ·107
42	7330383	8091672	46 21 ·712	88	1·5358897	4·0481254	231 56 ·416
43	7504916	8328406	47 43 ·095	89	1·5533430	4·7413488	271 39 ·557
44	7679449	8569026	49 5 ·814	90	1·5707963	∞	∞
45	·7853982	·8813736	50 29 ·939				